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SCHOOLS, ACADEMIES, AND COLLEGES.

BY

JAMES B. THOMSON, LL. D.,

AUTHOR OF A SERIES OF MATHEMATICS.

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PREFACE.

IT has long been a favorite plan of the author to make a Practical Algebra—a Book combining the important principles of the Science, with their application to methods of business.

Several years have elapsed since he began to gather and arrange materials for this object. Many of the more important parts have been written and re-written and again revised, till they have found embodiment in the book now offered to the public.

In the execution of this plan, clearness and brevity in the definitions and rules have been the constant aim.

A series of practical problems, applying the principles already explained, has been introduced into the fundamental rules, thus relieving the monotony of the abstract operations, and illustrating their use.

The principles are gradually developed, and explained in a manner calculated to lead the pupil to a full understanding of the difficulties of the science, before he is aware of their existence.

The rules are deduced from a careful analysis of practical problems involving the principles in question—a feature so extensively approved in the author's Series of Arithmetics.

The arrangement of subjects is consecutive and logical, their relation and mutual dependence being pointed out by frequent references.

The examples are numerous, and have been selected with a view to illustrate and familiarize the principles of the Science; while puzzles, calculated to waste the time and energy of the pupil, have been excluded.

Special attention has also been given to Factoring, Generalization, and the application of Algebra to business Formulas.

In these and other respects, it is believed, some advance is made beyond other books of the kind. While adapted to beginners, it covers as much ground as the majority of students master in their Academic and Collegiate course.

In presenting this book to the public, the author ventures to hope it may receive the approval so generously bestowed upon his former publications.

J. B. THOMSON.

BROOKLYN, N. Y., *Sept.*, 1877.

PUBLISHERS' NOTICE.

A KEY to this work, containing many valuable suggestions and full solutions, is now published for the use of Teachers only.

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ALGEBRA.

[illegible] 1878

CHAPTER I.

INTRODUCTION.

Art. 1. ***Algebra**** is the art of computing by letters and signs. These letters and signs are called *Symbols.*

2. ***Quantity*** is anything which can be measured; as distance, weight, time, number, &c.

3. A ***Measure*** of a quantity is a *unit* of that quantity established by law or custom, as the *Standard Unit.*

Thus, the measure of distance is the *yard;* of weight, the *Troy pound;* of time, the *mean solar day,* etc.

NOTATION.

4. ***Quantities*** in Algebra are expressed by *letters,* or by a combination of *letters and figures;* as, a, b, c, $3x$, $4y$, $5z$, etc.

The first letters of the alphabet are used to express *known* quantities; the last letters, those which are *unknown.*

QUESTIONS.—1. What is algebra? Letters and signs called? 2. Quantity? 3. A measure? 4. How are quantities expressed?

* From the Arabic *al* and *gabron,* reduction of parts to a whole.

5. The ***Letters*** employed have no ***fixed numerical value*** of themselves. *Any letter* may represent any number, and the same letter may represent *different* numbers, subject to one limitation; the *same* letter must always stand for the *same* number throughout the *same* problem.

6. The ***Relations*** of quantities, and the operations to be performed, are expressed by the same *signs* as in Arithmetic.

7. The ***Sign of Addition*** is a perpendicular cross, called *plus* (+). *

Thus, $a+b$ denotes the sum of a and b, and is read, "a plus b," or "a added to b."

8. The ***Sign of Subtraction*** is a short, horizontal line, called *minus* (—). †

Thus, $a-b$ shows that the quantity *after* the sign is to be subtracted from the one *before* it, and is read, "a minus b," or "a less b."

9. The ***Sign of Multiplication*** is an oblique cross (×).

Thus, $a \times b$ shows that a and b are to be multiplied together, and is read, "a times b," "a into b," or "a multiplied by b."

10. ***Multiplication*** is also denoted by a period between the factors; as, $a \cdot b$.

But the *multiplication of letters* is more commonly expressed by writing them together, the signs being omitted.

Thus, $5abc$ is equivalent to $5 \times a \times b \times c$.

11. The ***Sign of Division*** is a short, horizontal line between two dots (÷).

Thus, $a \div b$ shows that the quantity before the sign is to be divided by the one after it, and is read, "a divided by b."

---

5. Value of the letters? 6. Relations of quantities expressed? 7. Describe the sign of addition. 8. Subtraction. 9. Multiplication. 10. How else denoted?

* The Latin term *plus*, signifies *more*.

† The Latin *minus*, signifies *less*.

12. ***Division*** is also denoted by *writing the divisor under the dividend*, with a short line between them.

Thus, $\frac{a}{b}$ shows that a is to be divided by b, and is equivalent to $a \div b$.

13. The ***Sign of Equality*** is two short, parallel lines ($=$).

Thus, $a = b$ shows that the quantity before the sign is equal to the quantity after it, and is read, "a equals b," or "a is equal to b."

14. The ***Sign of Inequality*** is an acute angle, with the opening turned toward the greater quantity ($> <$).

Thus, $a > b$ shows that a is greater than b, and $a < b$ shows that a is less than b.

15. The ***Parenthesis*** ( ), or ***Vinculum*** ($\overline{\quad}$), indicates that the included quantities are taken *collectively*, or as *one* quantity.

Thus, $3(a + b)$ and $\overline{a + b} \times 3$, each denote that the sum of a and b is multiplied by 3.

16. The ***Double*** or ***Ambiguous Sign*** is a combination of the signs *plus* and *minus* ($\pm$).

Thus, $a \pm b$ shows that b is to be added to or subtracted from a, and is read, "a plus or minus b."

17. The ***character*** ($\therefore$), denotes *hence, therefore.*

18. ***Every quantity*** is supposed to be preceded by the sign *plus* or *minus*. When no sign is prefixed, the sign $+$ is always understood.

19. ***Like Signs*** are those which are all *plus*, or all *minus*; as, $+ a + b + c$, or $- x - y - z$.

20. ***Unlike Signs*** include both *plus* and *minus*; as, $a - b + c$ and $- x + y - z$.

---

11. Describe the sign of division? 12. How else denoted? 13. The sign of equality? 14. Of inequality? 15. Use of a parenthesis or vinculum? 16. Double sign? 17. Sign for "hence," etc.? 18. By what is every quantity preceded? When none is expressed, what is understood? 19. Like signs? 20. Unlike?

21. A ***Coefficient**** is a number or letter prefixed to a quantity, to show *how many times* the quantity is to be taken. Hence, a coefficient is a *multiplier* or *factor*.

Coefficients may be *numeral, literal,* or *mixed.*

Thus, in $5a$, 5 is a numeral coefficient of a; in bc, b is a literal coefficient of c; in $3dx$, $3d$ is a mixed coefficient of x.

When *no numeral* coefficient is expressed, 1 is always understood.

Thus, xy means $1xy$.

EXERCISES IN NOTATION.

22. To express a Statement by *Algebraic Symbols.*

It is required to express the following statement in algebraic symbols:

1. The product of a, b, and c, divided by the sum of c and d, is equal to the difference of x and y, increased by the product of a multiplied by 7.

Ans. $a \times b \times c \div (c + d) = (x - y) + 7a.$

Or $\dfrac{abc}{c + d} = (x - y) + 7a.$ Hence, the

RULE.—*For the words, substitute the signs which indicate the relations of the quantities and the operations to be performed.*

Express the following by algebraic symbols:

2. The sum of $4c$, d, and m, diminished by $5x$, equals the product of a and b.

3. The product of $5c$ and d, increased by the quotient of a divided by b, equals the product of x and y.

---

21. A coefficient? When no coefficient is expressed, what is understood?
22. How translate a statement from common language into algebraic symbols?

* *Coefficient,* Latin, *con,* with, and *efficere,* to effect; literally, a *co-operator.*

4. The quotient of $3b$ divided by $5c$, increased by $4m$, equals the sum of c and $6d$, diminished by the product of $7a$ and x.

5. If to the difference between a and b, we add the product of x into y, the sum will be equal to m multiplied by $6n$.

6. The difference between x and y, added to the sum of $4a$ and b minus m, equals the product of c and d, increased by 15 times m.

☞ **These and the following exercises should be supplemented by dictation, until the learner becomes familiar with them.**

23. To translate *Algebraic Expressions* into Common Language.

Express the following statement in common language:

1. $\frac{a+b}{d} = 2abc - (x+y) + \frac{d}{ab}.$

Substituting words for signs, we have the sum of a and b, divided by d, equals twice the product of a, b, and c, diminished by the sum of x and y, increased by the quotient of d divided by the product of a and b, *Ans.* Hence, the

RULE.—*For the signs indicating the given relations and operations, substitute words.*

Express the following in common language:

2. $\frac{2ab}{x} + a - b = \frac{a+b}{c} + axy - 4cd.$

3. $\frac{3b+c}{8} + 3x = \frac{3cd}{a} + xyz - \frac{c}{d}.$

4. $\frac{3a}{5} - ax + bc = \frac{4a-b}{x} + \frac{cd}{4} - 3x.$

5. $\frac{abc-x}{3d} \pm 3x + 5y = \frac{cdh+x}{2a} - xy.$

6. $\frac{4axy}{5a} + \frac{a-b}{x} = \frac{x+y}{a} - \frac{2a+d}{3c}.$

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23. **How translate algebraic expressions into common language?**

ALGEBRAIC OPERATIONS.

24. An ***Algebraic Operation*** is a combination of quantities according to the principles of algebra, setting down the results in form.

25. A ***Problem*** is something proposed to be done, as a question to be solved.

The ***Solution*** of a problem is finding the answer.

26. The ***Equality*** between two quantities is denoted by the sign $=$. (Art. 13.)

27. The ***Expression of Equality*** between two quantities is called an ***Equation.*** Thus, $15 - 3 = 7 + 5$ is an equation.

PROBLEMS.

28. The following problems are solved by combining the preceding principles with those of Arithmetic.

1. A and B found a purse containing 12 dollars, and divided it in such a manner that B's share was three times as much as A's. How many dollars did each have?

BY ARITHMETIC.—A had 1 share and B 3 shares; now 1 share + 3 shares are 4 shares, which are equal to 12 dollars. If 4 shares equal 12 dollars, 1 share is equal to as many dollars as 4 is contained times in 12, which is 3. Therefore, A had 3 dollars, and B had 3 times as much, or 9 dollars.

OPERATION.

Let	$x =$	A's share,
then	$3x =$	B's share,
and	$x + 3x =$	12 dollars,
that is,	$4x =$	12 dollars.
Hence,	$x =$	3 dol., A.
	$3x =$	9 dol., B.

BY ALGEBRA.—We represent A's share by x, and form an equation by treating this letter as we treat the answer in proving an operation. If x represent A's share, $3x$ will represent B's, and $x + 3x = 12$ dollars, the sum of both. Uniting the terms, we have the equation, $4x = 12$ dollars. To remove the coefficient 4, we

24. What is an algebraic operation? 25. A problem? A solution? 26. Equality denoted? 27. The expression of equality called?

divide both sides of the equation by it. For, if equals are divided by equals, the quotients are equal. Therefore, $x = 3$ dollars, A's share, and $3x = 9$ dollars, B's share. (Ax. 5.)

PROOF.—By the first condition, 9 dollars, B's share = 3 times 3 dollars, A's share. By the second condition, 9 dollars + 3 dollars = 12 dollars, the sum found. Hence,

29. When a quantity on either side of the equation has a *coefficient*, that coefficient may be removed, *by dividing every term on both sides of the equation by it.*

2. A and B together have 15 pears, and A has twice as many as B: how many has each?

BY ALGEBRA.—If x represents B's number, $2x$ will represent A's, and $x+2x$, or $3x$, will represent the number of both. Dividing both sides by the coefficient 3, we have $x = 5$ pears, B's number, and $2x = 10$ pears, A's.

OPERATION.

Let	$x =$	B's number;
then	$2x =$	A's "
and	$3x =$	15 pears.
$\therefore$	$x =$	5 pears, B's.
	$2x =$	10 pears, A's.

NOTE.—It is advisable for the learner to solve each of the following problems by Arithmetic and by Algebra.

3. A lad bought an apple and an orange for 8 cents, paying 3 times as much for the orange as for the apple. What was the price of each?

4. A farmer sold a cow and a ton of hay for 40 dollars, the cow being worth 4 times as much as the hay. What was the value of each?

5. The sum of two numbers is 36, one of which is 3 times the other. What are the numbers?

6. A, B, and C have 28 peaches; B has twice as many as C, and A twice as many as B. How many has each?

7. A father is 3 times the age of his son, and the sum of their ages is 48 years. How old is each?

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29. How remove a coefficient?

8. A and B trade in company, and gain 100 dollars. If A puts in 4 times as much as B, what will be the gain of each?

9. The sum of three numbers is 90. The second is twice the first, and the third as many as the first and second: what are the numbers?

10. A cow and calf were sold for 63 dollars, the cow being worth 8 times as much as the calf. What was the value of each?

11. A man being asked the price of his horse, replied that his horse, saddle and bridle together were worth 126 dollars; that the saddle was worth twice as much as the bridle, and the horse 7 times as much as both the others. What was each worth?

12. A man bequeathed $36,000 to his wife, son, and daughter, giving the son twice as much as the daughter, and the wife 3 times as much as the son and daughter. What did each receive?

13. The sum of three numbers is 1877; the second is 3 times the first, and the third exceeds the other two by 5. What are the numbers?

POWERS AND ROOTS.

30. A ***Power*** is the product of two or more *equal* factors.

Thus, the product 2×2, is the *square* or *second* power of 2; $x \times x \times x$ is the *cube* or *third* power of x.

31. The ***Index*** or ***Exponent*** of a *power* is a figure or letter placed at the right, above the quantity.

Thus, a^1 denotes a, or the *first* power.
a^2 " $a \times a$, the *square*, or *second* power.
a^3 " $a \times a \times a$, the *cube*, or *third* power, etc.

32. A ***Root*** is one of the *equal factors* of a quantity.

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30. What is a power? 31. How denoted?

33. ***Roots*** are denoted by the ***Radical Sign*** ($\sqrt{\ }$) prefixed to the quantity, or by a *fractional exponent* placed after it.

Thus, $\sqrt{a}$, $a^{\frac{1}{2}}$, or $\sqrt[2]{a}$ denote the square root of the quantity a; $\sqrt[3]{a}$ shows that the cube root of a is to be extracted, etc.

34. The ***Index of the Root*** is the figure placed over the radical sign. The index of the square root is usually omitted.

(For negative indices, see Arts. 256, 258.)

Read the following examples:

1. $a^2 + 3a$.	7. $4(a-b)^2$.
2. $b^3 - c^2$.	8. $a^2 + 2ab + b^2$.
3. $a + b^2 - c$.	9. $\sqrt{a+b}$.
4. $x^3 - y + y^2$.	10. $\sqrt[3]{a^2 - b^2}$.
5. $2y^2 + z^3 - z$.	11. $2a^{\frac{1}{2}} + c^{\frac{1}{3}}$.
6. $3(a^2 + b)$.	12. $4x^{\frac{1}{3}} + 2y^{\frac{1}{4}}$.

Write the following in algebraic language:

13. The square of a plus the square of b.
14. The square of the sum of a plus b.
15. The sum of a and b, minus the square of c.
16. The square root of a plus the square root of x.
17. The cube root of x minus the fifth power of y.
18. The cube root of a plus the square of b.

ALGEBRAIC EXPRESSIONS.

35. An ***Algebraic Expression*** is any quantity expressed in algebraic language; as, $3a$, $5a - 7b$, etc.

36. The ***Terms*** of an algebraic expression are those parts which are connected by the signs $+$ and $-$.

Thus, in $a+b$, there are two terms; in $x+y\times z-a$ there are three.

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32. A root? 33. How denoted? 34. What is the figure placed over the radical sign called? 35. What is an algebraic expression? 36. Its terms?

NOTE.—Letters combined by the signs $\times$ or $\div$ do not constitute separate terms. Such a combination, to form a term, must have the sign $+$ or $-$ prefixed to it, and the operations indicated by the signs $\times$ or $\div$ must be performed before the terms can be added to or subtracted from the preceding term. (Art. 36.) Thus, $a+b\times c$ has two terms, $b\times c$ forming one term and a the other.

37. The ***Dimensions*** of a term are its several *literal* factors.

38. The ***Degree*** of a term depends on the number of its *literal factors*, and is always equal to the *sum* of their exponents.

Thus, ab contains *two* factors, a and b, and is of the *second* degree. a^2x contains *three* factors, a, a, and x, and is of the *third* degree. b^2x^3 contains *five* factors, b, b, x, x, x, and is of the *fifth* degree.

39. The ***Numerical Value*** of an algebraic expression is the *number* which it represents when its terms are combined as indicated by the signs. (Art. 36.)

40. To Find the *Numerical Value* of an algebraic expression.

1. If $a=5$, $b=7$, and $x=9$, what is the value of $6a+8b+3x$?

ANALYSIS. — Since $a=5$, $6a$ must equal 6 times 5, or 30; since $b=7$, $8b$ must equal 8 times 7, or 56; and since $x=9$, $3x$ must equal 3×9, or 27. Now $30+56+27=113$. Therefore, the value of the given expression is 113. Hence, the

OPERATION.

$$\begin{aligned} 6a &= 5\times 6 = 30 \\ 8b &= 7\times 8 = 56 \\ 3x &= 3\times 9 = \underline{27} \\ & \textit{Ans. } 113 \end{aligned}$$

RULE.—*For the letters, substitute the figures which the letters represent, and perform the operations indicated by the signs.*

2. If $b=3$, $c=5$, and $d=8$, what is the value of $5b+7c+6d$?

SUGGESTION. $15+35+48=98$, *Ans.*

37. The dimensions of a term? 38. Degree? 39. Numerical value of an algebraic expression? 40. How found?

Find the numerical value of the following expressions, when $a = 2$, $b = 3$, $c = 4$, $d = 5$, and $x = 6$.

3. $4a + 6ab + 5c =$ how many? *Ans.* 64.
4. $(a + b) \times c \times d - x \div c =$ how many?
5. $(x - a) + ax + c \div a =$ how many?
6. $x \div 2 + (d - c) + bc - x =$ how many?
7. $dx + (c - a) \times (a - b) + x =$ how many?
8. $d + x \times (c - a) + a - x + c =$ how many?

CLASSIFICATION OF ALGEBRAIC QUANTITIES.

41. Quantities in Algebra are primarily divided into *known* and *unknown*.

42. A ***Known Quantity*** is one whose value is given.

An ***Unknown Quantity*** is one whose value is not given.

These quantities are subdivided into like and unlike, positive and negative, simple, compound, monomials, etc.

43. ***Like Quantities*** are those which are expressed by the *same power* of the *same letters;* as, a and $2a$, $2x^2$ and x^2.

44. ***Unlike Quantities*** are those which are expressed by different letters, or by different powers of the *same letters;* as $2x$ and $3y$, $2x$ and x^2.

NOTE.—An exception must be made in cases where letters are regarded as coefficients. Thus, ax^2 and bx^2 are like quantities, when a and b are considered coefficients.

45. A ***Positive Quantity*** is one that is to be *added*, and has the sign + prefixed to it; as, $4a + 3b$.

46. A ***Negative Quantity*** is one that is to be *subtracted*, and has the sign — prefixed to it; as, $4a - 3b$.

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41. How are quantities in Algebra primarily divided? 42. A known quantity? Unknown? 43. Like quantities? 44. Unlike? 45. A positive quantity? 46. A negative?

47. The terms ***Positive*** and ***Negative*** denote oppositeness, either in the nature of the quantities to which they are applied, or in the application of those quantities. The former denotes a quantity to be *added*, the latter one to be *subtracted.* Thus, if a man's gains are *positive*, his losses are *negative.* If distances north of the equator are *positive*, those south are *negative.*

48. A ***Simple Quantity*** is a single letter, or several letters written together without the sign $+$ or $-$; as, a, ab, $3xy$.

49. A ***Compound Quantity*** is two or more simple quantities connected by the sign $+$ or $-$; as $3a + 4b$, $2x - y$.

50. A ***Monomial**** has but one term; as, a, $3b$.

51. A ***Binomial*** † has *two terms;* as, $a + b$, $a - b$.

NOTES.—1. The expression $a - b$ is often called a *residual*, because it denotes that which remains after a part is subtracted.

2. A *binomial* is sometimes called a *polynomial.*

52. A ***Trinomial*** ‡ has three terms; as, $a + b - c$.

53. A ***Polynomial*** ‖ has three or more terms; as, $a + b - c + x$.

54. An ***Homogeneous Polynomial*** has all its terms of the same degree.

Thus, $2ab + cd + 5xy$ is homogeneous; but $4abc + c^2 + 5x$ is not.

55. The ***Reciprocal*** of a quantity is a unit divided by that quantity.

Thus, the reciprocal of a is $\frac{1}{a}$; the reciprocal of $a+b$ is $\frac{1}{a+b}$.

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47. What do the terms positive and negative denote? 48. A simple quantity? 49. A compound? 50. A monomial? 51. A binomial? *Note.* The expression $a-b$ called? 52. A trinomial? 53. A polynomial? 54. When homogeneous? 55. The reciprocal of a quantity?

* Greek, *monos*, single, and *nomē*, term, having *one term.*

† Latin, *bis*, two, and *nomen*, name, having *two terms.*

‡ Latin, *tres*, three, and *nomen*, name, having *three terms.*

‖ Greek, *polus*, many, and *onoma*, name, having *many terms.*

FORCE OF THE SIGNS.

56. ***Each term*** of an algebraic expression is preceded by the sign + or —, expressed or understood. (Art. 18.)

The ***Force*** of each of these signs is limited to the *term* which follows it; as, $7 + 5 - 3 = 12 - 3 = 9$; $15 - 6 + 8 = 9 + 8 = 17$.

57. If a term, preceded by the sign + or —, is combined with other letters by the sign $\times$ or $\div$, each of these letters forms a part of that term, and the operations indicated, taken in their order, must be performed before any part of the term can be *added* to or *subtracted* from any other term.

Thus, the expression $12 + 4 \times 2$, shows that 4 is to be multiplied by 2 and the product added to 12, and is equal to 20.

In like manner, the expression $16 - 8 \div 2$, shows that 8 is to be divided by 2 and the quotient subtracted from 16, and is equal to 12.

58. If two or more terms joined by + or — are to be subjected to the *same operation*, they must be connected by a *parenthesis* or *vinculum*.

Thus, if $a + b$ or $a - b$ is to be multiplied or divided by c, the operations are indicated by $(a+b) \times c$, or $c(a+b)$; $(a - b) \div c$, or $\frac{a-b}{c}$.

EXERCISES.

1. $50 + 5 \times 2 =$ what number?
2. $50 - 5 \times 2 =$ what number?
3. $ac + 4b \times 2 =$ what number?
4. $5b - 6d \div 3 =$ what number?
5. $15 + 5 \times 3 + 10 \div 2 =$ what?
6. $18 - 2 \times 4 \div 2 + 10 =$ what?
7. $3x + 8y \div 4 + a \times b =$ what?
8. $6b - 7c \times x + 9a \div 3 =$ what?
9. $(b + c) \times xy =$ what?

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56. By what are all algebraic terms preceded? The force of each of these signs? 57. Of the signs $\times$ and $\div$? 58. Of the parenthesis and vinculum?

10. $3x \times 5y \div 2z + a =$ what?

11. $(b - a) \div xy + 2z =$ what?

12. $3x + xy + 2z \times 3y =$ what?

13. The difference of x and y into a less b divided by d = what number?

Find the value of the following expressions, in which $a = 3$, $b = 4$, $c = 2$, $x = 6$, $y = 8$, and $z = 10$:

14. $a + (a \times x) \div c + y \times z =$ what?

15. $2b \div (x - b) + a \times b \times y + 2z =$ what?

AXIOMS.

59. An ***Axiom*** is a self-evident truth.

1. Things which are equal to the *same* thing, are equal to each other.
2. If equals are *added* to equals, the *sums* are equal.
3. If equals are *subtracted* from equals, the *remainders* are equal.
4. If equals are *multiplied* by equals, the *products* are equal.
5. If equals are *divided* by equals, the *quotients* are equal.
6. If a quantity is *multiplied* and *divided* by the same quantity, its *value* is not altered.
7. If the same quantity is *added to* and *subtracted from* another quantity, the *value* of the latter is not altered.
8. The *whole* is greater than its *part*.
9. The *whole* is equal to the sum of *all its parts*.
10. Like *powers* and like *roots* of equal quantities, are *equal*.

NOTE.—The importance of thoroughly understanding the *definitions* and *principles* cannot be too deeply impressed upon the mind of the learner. The questions at the foot of the page are designed to direct his attention to the more important points. Teachers, of course, will not be confined to them.

CHAPTER II.

ADDITION.

60. ***Addition*** in Algebra is uniting two or more quantities and *reducing* them to the simplest form.

61. The ***Result*** is called the ***Sum*** or ***Amount.***

62. ***Quantities*** expressed by *letters* are regarded as *concrete* quantities. Hence, their coefficients may be added, subtracted, multiplied, and divided like concrete numbers.

Thus, $3a$ and $4a$ are $7a$, $4b$ and $5b$ are $9b$, as truly as 3 apples and 4 apples are 7 apples, or as 4 bushels and 5 bushels are 9 bushels.

PRINCIPLES.*

63. 1°. *Like quantities only can be united in one term.*

2°. *The sum of two or more quantities is the same in whatever order they are added.*

CASE I.

64. To Add like Monomials which have *like* signs.

1. What is the sum of $15ab + 13ab + 19ab$?

ANALYSIS.—These terms are like quantities and have like signs. (Art. 19.) We therefore add the coefficients, to the sum annex the common letters, and prefix the common sign. The result, $+ 47ab$, is the answer required.

OPERATION.

$$\begin{array}{r} +\ 15ab \\ +\ 13ab \\ +\ 19ab \\ \hline +\ 47ab, \end{array}\ \textit{Ans.}$$

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60. What is addition? 61. The result called? 62. How are quantities expressed by letters regarded? 63. First principle? Second?

* The expressions 1°, 2°, 3°, etc., denote *first, second, third,* etc.

2. What is the sum of $-14xy$, $-16xy$, and $-18xy$?

ANALYSIS.—Since these terms are like quantities, and have like signs, we add them as before, and prefix the sign $-$ to the result, for the reason that all the quantities have the sign $-$. Hence, the

$$\begin{array}{r} -14xy \\ -16xy \\ -18xy \\ \hline -48xy, \textit{ Ans.} \end{array}$$

RULE.—*Add the coefficients; to the sum annex the common letters, and prefix the common sign.*

(3.)	(4.)	(5.)	(6.)	(7.)
$3ab$	$5xy$	$7a^2$	$-7bcd$	$-4x^3y^2$
$5ab$	$8xy$	$3a^2$	$-3bcd$	$-3x^3y^2$
$6ab$	xy	$4a^2$	$-5bcd$	$-x^3y^2$
$7ab$	$3xy$	a^2	$-8bcd$	$-8x^3y^2$

8. Add $5ab^2 + 17ab^2 + 23ab^2$.

9. Add $-8abx^2y^2 - 3abx^2y^2 - 28abx^2y^2$.

10. Add $5b^2dm^3 + 7b^2dm^3 + 9b^2dm^3 + 8b^2dm^3$.

11. If $3a + 5a + a + 7a = 48$, to what is a equal?

SOLUTION. $3a+5a+a+7a=16a$; hence, $a=48 \div 16$, or 3. *Ans.*

12. If $4bc + 9bc + 2bc + 5bc = 80$, to what is bc equal?

13. If $xy + 3xy + 5xy + 4xy = 65$, to what is xy equal?

CASE II.

65. To Add like Monomials which have *Unlike* signs.

14. What is the sum of $5ab - 3ab - 7ab + 9ab + 6ab - 8ab$?

ANALYSIS.—For convenience in adding, we write the *negative* terms one under another in the right-hand column, with the sign $-$ before each, and the *positive* terms in the next column on the left. We then find the sum of the coefficients of the positive and negative

OPERATION.

$$\begin{array}{rr} 5ab & -\ 3ab \\ 9ab & -\ 7ab \\ 6ab & -\ 8ab \\ \hline 20ab & -\ 18ab = 2ab, \textit{ Ans.} \end{array}$$

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64. How add monomials which have like signs?

terms separately; and taking the less sum from the greater, the result $2ab$, is the answer. Hence, the

RULE.—I. *Write the positive and negative terms in separate columns with their proper signs, and find the sum of the coefficients of each column separately.*

II. *From the greater subtract the less; to the remainder prefix the sign of the greater, and annex the common letters.*

NOTE—If two equal quantities have *opposite* signs, they *balance* each other, and may be omitted.

15. Add $4d + 3d - 5d + 6d - 2d$. *Ans.* $6d$.
16. Add $-5x + 6x + 8x - 3x + 9x - 7x$.
17. Add $3abc + 12abc - 6abc + 5abc - 10abc - 3abc$.
18. Add $2b - 5b + 4b - 6b - 7b$.
19. Add $-6y + 4y - 8y - 9y + 8y - y$.
20. Add $4m + 16m - 8m - 9m + 5m - 10m$.
21. If $6ab + 14ab - 7ab + 15ab - 12ab + 16ab = 32$, to what is ab equal?
22. To what is bcd equal, if $bcd - 3bcd + 4bcd + 4bcd - 5bcd = 75$?

REMARK.—The *sum* in Arithmetic is *always greater* than any of its parts. But, in Algebra, it will be observed, the *sum* of a *positive* and *negative* quantity is always *less* than the *positive* quantity. It is thence called ***Algebraic Sum.***

66. ***Unlike Quantities*** cannot be united in *one term.* Their *sum* is indicated by writing them one after another, with their proper signs. (Art. 63, Prin. 1.)

Thus, the sum of $7g$ and $3d$ is neither $10g$ nor $10d$, any more than 7 guineas and 3 dollars are 10 guineas or 10 dollars. Their sum is $7g + 3d$. (Art. 63, Prin. 1.)

67. ***Polynomials*** are added by uniting like quantities, as in adding monomials.

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65. How add monomials having unlike signs? *Rem.* What is true of the sum in Arithmetic? In Algebra? 66. How add unlike quantities? 67. Polynomials?

23. What is the sum of the polynomial $3ab - 3b + 4d - 3x$; $-5ab + 4x - c - 2d$; and $bg + d + 2ab + b$?

ANALYSIS.—For convenience, we write the quantities so that like terms shall stand one under another, and uniting those which are alike, the result is $-2b+3d+x+bg-c$.

OPERATION.

$$\begin{array}{rrrrrr} 3ab & -\ 3b & +\ 4d & -\ 3x & & -\ c \\ -\ 5ab & +\ \ b & -\ 2d & +\ 4x & & \\ 2ab & & +\ \ d & & +\ bg & \\ \hline & -\ 2b & +\ 3d & +\ \ x & +\ bg & -\ c, \end{array} \quad Ans.$$

68. From the preceding illustrations and principles we deduce the following

GENERAL RULE.

I. *Write the given quantities so that like terms shall stand one under another.*

II. *Unite the terms which are alike, and to the result annex the unlike terms with their proper signs.*

1. Add $5a - 3a + 6a + 7a + 9a + 2b - 3d$.
2. Add $8mn + 3mn - 4mn + 9mn - xy + bc$.
3. Add $3bc - 7bc + xy - mn + 11bc + 9bc$.
4. Add $5ab - 3mn - ab + 3ab + 2z - 4ab + ab$.
5. Add $3xy - xy + ab - 7xy + b + 8xy - xy + 13xy$.

69. ***Compound Quantities*** inclosed in a parenthesis, are taken *collectively*, or as *one* quantity. Hence, if the quantities are *alike*, their *coefficients* and *exponents* are treated as the coefficients and exponents of like monomials. (Art. 64.)

6. What is the sum of $3(a+b) + 5(a+b) + 7(a+b)$?

SOLUTION. $3(a+b)$ and $5(a+b)$ and $7(a+b)$ are $15(a+b)$. *Ans.*

7. Add $13(a+b) + 15(a+b) - 7(a+b)$.
8. Add $8c(x-y) + 7c(x-y) - 5c(x-y) + 9c(x-y)$.
9. Add $3a\sqrt{xy} + 5a\sqrt{xy} - 7a\sqrt{xy} + 6a\sqrt{xy}$.
10. Add $5\sqrt{a} + 3\sqrt{a} - 8\sqrt{a} + 9\sqrt{a} - 3\sqrt{a}$.
11. Add $8\sqrt{x-y} - 3\sqrt{x-y} + 5\sqrt{x-y}$.

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68. The general rule for addition? 69. How add quantities included in a parenthesis?

70. The *sum* of *unlike* quantities having a *common* letter or letters, may be expressed by *inclosing the other letters*, with their signs and coefficients, in a parenthesis, and *annexing* or *prefixing* the *common* letter or letters to the result.

12. What is the sum of $5ax + 3bx - 4cx$?

SOLUTION. $5ax + 3bx - 4cx = (5a + 3b - 4c)x$, or $x(5a + 3b - 4c)$. *Ans.*

13. Add $7a - 6ba + 3da - 3ma$.
14. Add $aby + 3y - 2cy - 5my$.
15. Add $9m + abm - 7cm + 3dm$.
16. Add $13ax - 3bx + cx - 3dx + mx$.
17. Add $axy + bxy - cxy$.

PROBLEMS.

71. Problems requiring equal quantities to be added to each side of the equation.

1. A has 3 times as many marbles as B, lacking 6; and both together have 58. How many has each?

ANALYSIS.—If x represents B's number, then will $3x-6$ represent A's, and $x+3x-6 = 58$, the sum of both. To remove -6, we add an *equal positive* quantity to each side of the equation. (Axiom 2.) Uniting the terms, we have $4x = 64$, and $x = 16$, B's, and 3 times $16-6$, or $42 = $ A's No.

OPERATION.

Let $x = $ B's No.;
then $3x - 6 = $ A's "
$x + 3x - 6 = 58$, both.
$x + 3x - 6 + 6 = 58 + 6$
$4x = 64$
$x = 16$, B's No.
$3x - 6 = 42$, A's "

72. When a *negative* quantity occurs on either side of an equation, that quantity may be removed by *adding an equal positive quantity to both sides.*

NOTE.—In forming the equation, we treat x as we do the answer in proving an operation.

2. A kite and a ball together cost 46 cents, and the kite cost 2 cents less than twice the cost of the ball. What was the cost of each?

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70. How may the sum of unlike quantities which have a common letter be expressed?

3. In a basket there are 75 peaches and pears; the number of pears being double that of the peaches, wanting 3. How many are there of each?

4. The sum of two numbers is 85, and the greater is 5 times the less, wanting 5. What are the numbers?

5. A certain school contains 40 pupils, and there are twice as many girls, lacking 5, as boys. How many are there of each?

6. If $44x + 65x - 24 = 85$, what is the value of x?

7. If $7x - 3 + 2x = 60$, what is the value of x?

8. If $4y + 2y + 5y - 7 = 70$, what is the value of y?

9. The whole number of votes cast for A and B at a certain election was 450; A had 20 votes less than 4 times the number for B. How many votes had each?

10. The sum of two numbers is 177; the greater is 3 less than 4 times the smaller. What are the numbers?

11. What is the value of y, if $4y + 3y + 2y - 12 = 60$?

12. A lad bought a top and a ball for 32 cents; the price of the ball was 3 times that of the top, minus 4 cents. What was the price of each?

13. A man being asked the price of his saddle and bridle, replied that both together cost 40 dollars, the former being 4 times the price of the latter, minus 5 dollars. What was the price of each?

14. A lad spent a dollar during a holiday, using three times as much of it in the afternoon as in the morning, minus 4 cents; how much did he spend in each part of the day?

Find the value of x in the following equations:

15. $3x + 6x + 4x + 5x - 8 = 154$. *Ans.* 9.

16. $2x + 5x + 3x - 10 = 130$.

17. $4x + 3x + 7x - 12 = 86$.

18. $10x - 4x + 9x - 25 = 155$.

19. $15x - 7x - 2x - 60 = 300$.

20. $18x - 4x + x - 75 = 225$.

CHAPTER III.

SUBTRACTION.

73. ***Subtraction*** is finding the *difference* between two quantities.

The ***Minuend*** is the quantity from which the subtraction is made.

The ***Subtrahend*** is the quantity to be subtracted.

The ***Difference*** is the result found by subtraction.

74. Since quantities expressed by letters are regarded as *concrete*, the *coefficient* of one letter may be subtracted from that of another, like concrete numbers. (Art. 62.)

Thus, $7a - 3a = 4a$; $8b - 5b = 3b$.

PRINCIPLES.

75. 1°. *Like quantities only can be subtracted one from another.*

2°. *The sum of the difference and subtrahend is equal to the minuend.*

3°. *Subtracting a positive quantity is equivalent to adding an equal negative one.*

Thus, let it be required to subtract $+4$ from $6+4$.

Taking $+4$ from $6+4$, leaves 6.
Adding -4 to $6+4$, we have $6+4-4$.
But (Ax. 7) $6+4-4$ is equal to 6.

4°. *Subtracting a negative quantity is the same as adding an equal positive one.*

---

73. Define subtraction. The Minuend. Subtrahend. Difference. 75. Name the first principle. Second. Illustrate Prin. 3 upon the blackboard. Illustrate Prin. 4.

Thus, let it be required to subtract -4 from $10-4$.

Taking -4 from $10-4$, leaves 10.
Adding $+4$ to $10-4$, we have $10-4+4$.
But (Ax. 7) $10-4+4$ is equal to 10.

Again, if the assets of an estate be \$500, and the liabilities \$300, the former being considered *positive* and the latter *negative*, the net value of the estate will be \$500 − \$300 = \$200. *Taking* \$50 from the *assets* has the same effect on the *balance* as *adding* \$50 to the *liabilities*. In like manner, taking \$50 from the *liabilities* has the same effect as adding \$50 to the *assets*.

76. To Find the *Difference* between two like Quantities.

This proposition includes three classes of examples, as will be seen in the following illustrations:

1. From $25a$ subtract $17a$.

REMARK.—1. In this example the *signs* are *alike*, and the subtrahend is *less* than the minuend. Subtracting a positive quantity is equivalent to adding an *equal negative* one. (Prin. 3.) We therefore change the sign of the subtrahend, and then unite the terms as in addition. Thus, $25a-17a=8a$.

OPERATION.

$25a$	Minuend.
$-17a$	Subtrahend.
$8a$	Difference.

2. From $4a$ subtract $7a$.

REMARK.—2. In this example the signs are *alike*, but the subtrahend is *greater* than the minuend. Changing the sign of the subtrahend, and uniting the terms as before, the subtrahend cancels the minuend, and has $-3a$ left. (Prin. 3.)

OPERATION.

$4a$	Minuend.
$-7a$	Subtrahend.
$-3a$	Difference.

3. From $45ab$ subtract $-29ab$.

REMARK.—3. In this example the signs are *unlike*. Subtracting a *negative* quantity is the same as adding an *equal positive* one. (Prin. 4.) Changing the sign of the subtrahend and proceeding as before, we have $45ab+29ab=74ab$. *Ans.*

OPERATION.

$45ab$	Minuend.
$+29ab$	Subtrahend.
$74ab$, *Ans.*	

---

76. How find the difference between two like quantities?

4. From $9bc + 7d - 5x$, take $3bc + 2d - 4x$.

ANALYSIS. — In subtraction of polynomials, for *convenience*, we place like terms under each other. Then, changing the signs of all the terms in the subtrahend, we unite them as before.

OPERATION.

$$\begin{array}{r} 9bc + 7d - 5x \\ -3bc - 2d + 4x \\ \hline 6bc + 5d - x, \end{array} \textit{ Ans.}$$

77. From the preceding illustrations and principles we deduce the following

GENERAL RULE.

I. *Write the subtrahend under the minuend, placing like terms one under another.*

II. *Change the signs of all the terms of the subtrahend, or suppose them to be changed, from + to —, or from — to +, and then proceed as in addition.* (Art. 74, Prin. 3, 4.)

NOTES.—1. *Unlike quantities* can be subtracted only by *changing the signs* of all the terms of the subtrahend, and then writing them after the minuend. (Art. 66.)

2. As soon as the student becomes familiar with the principles of subtraction, instead of actually changing the signs of the subtrahend, he may simply suppose them to be changed.

EXAMPLES.

1. From $43c + d$, take $25c + d$. *Ans.* $18c$.
2. From $49x$, take $23x + 3$. *Ans.* $26x - 3$.
3. From $28xyz$, take $14xyz$.
4. From $-43ab$, take $+19ab$.
5. From $4ab$, take $-15ab$.
6. From $43xy$, take $+16xy$.

	(7.)	(8.)	(9.)	(10.)
From	$20ac$	$42ax^2$	$37a^2b$	$-29x^2y^2$
Take	$-23ac$	$5ax^2$	$-14a^2b$	$+15x^2y^2$

77. General rule for subtraction? *Note.* How subtract unlike quantities.

	(11.)	(12.)	(13.)	(14.)
From	$31a^2b$	$19abx^3$	$-33m^2x$	$41x^3y$
Take	$-7a^2b$	$19abx^3$	$+44m^2x$	$-12x^3y$

15. A is worth \$100, and B owes \$50 more than he is worth. What is the difference in their pecuniary standing?

16. What is the difference in temperature, when the thermometer stands 15 degrees *above* zero, and when at 10 degrees below?

17. By speculation, A gained on a certain day \$275, and B lost \$145. What was the difference in the results of their operations?

	(18.)	(19.)	(20.)
From	$7xy - 8a$	$8b^2 + 7am$	$13x^2 - 7y^2$
Take	$3xy - 2a$	$-5b^2 - 9am$	$-5x^2 - 8y^2 - 6a$

21. From $13ab + d - x$, subtract $5m - 3n$.
22. From $9cd - ab$, take $2m - 3n - 4y$.
23. From $13m - 15$, take $-5m + 8$.
24. From $7x^2 - 5x + 15$, take $-5x^2 + 8x + 15$.
25. From $19ab - 2c - 7d$, take $3ab - 15c - 8d$.
26. From a, take $b - c$, and prove the work.
27. From $11(a + b)$, take $5(a + b)$.
28. From $17(a - b + x)$, take $8(a - b + x)$.
29. Subtract $-18(a + b)$ from $-13(a + b)$.
30. Subtract $21(x^2 - y)$ from $14(x^2 - y)$.

31. A and B formed an equal partnership and made \$1,000. B's share by right was \$1,000 — \$500; but wishing to withdraw, he agreed to subtract \$100 from his share. What would A's share be?

32. What is the difference of longitude between two places, one of which is 23 degrees due east from the meridian of Washington, the other 37 degrees due west?

REMARK.—The *subtrahend*, in Algebra, is often *greater* than the *minuend*, and the *difference* between a *positive* and *negative* quantity *greater* than either of them. It is thence called ***Algebraic Difference.***

78. The ***Difference*** of unlike quantities which have a common letter or letters may be indicated by *enclosing all the other letters,* with their coefficients and signs, in a parenthesis, and *annexing,* or *prefixing* the common letter or letters to the result.

33. From $3am$, take $2bm$.

ANALYSIS. $3am = 3a$ times m, and $2bm = 2b$ times m; therefore, $3am - 2bm = (3a - 2b)\,m$, or $m\,(3a - 2b)$. *Ans.*

34. From $2bx^2$, take $cx^2 - dx^2$.
35. From aby, take $cy + dy - xy$.
36. From $7a^2$, take $ba^2 - ca^2$.
37. From abx, take $3cx + dx + mx$.
38. From $8xy$, take $abxy - cxy + dxy$.
39. From $5ac + bmc$, take $3ac - dc$.

APPLICATIONS OF THE PARENTHESIS.

79. A *parenthesis,* we have seen, shows that the quantities inclosed by it are taken *collectively,* and subjected to the *operation* indicated by the *sign* which precedes it. (Art. 15.)

80. A parenthesis having the sign + prefixed to it, may be removed from an expression, if the *signs* of the included terms remain *unchanged.*

Thus, $a - b + (c - d + e) = a - b + c - d + e$. Hence,

81. Any number of terms may be inclosed in a parenthesis and the sign + placed before it, if the *signs* of the inclosed terms remain *unchanged.*

Thus, $a + b - c + d = a + (b - c + d)$, or $a + b + (-c + d)$.

NOTE.—This principle affords a convenient method of indicating the *addition* of polynomials. (Art. 67.)

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78. How subtract unlike quantities having a common letter or letters? 79. What is the object of a parenthesis? 80. How removed when the sign + is prefixed to it.

82. A parenthesis having the sign $-$ prefixed to it, may be removed by *changing the signs* of all the inclosed terms from $+$ to $-$ and $-$ to $+$.

Thus, removing it from the equal expressions,

$$\left.\begin{matrix} a - b - (-c + d) \\ a - b - (d - c) \end{matrix}\right\} = a - b + c - d. \quad \text{Hence,}$$

83. Any number of terms may be inclosed by a parenthesis, and the sign $-$ placed before it, if all the signs of the inclosed terms are changed.

Thus, $a-b+c-d = a-(b-c+d)$, or $a-b-(-c+d)$, etc.

NOTE.—This principle enables us to express a polynomial in different forms without changing its value.

1. How express $a - x + c$, using a parenthesis?
   Ans. $a - x + c = a - (x - c)$, or $a - (-c + x)$.
2. How express $a - b - x - y + z$, using a parenthesis?
   Ans. $a - b - (x + y - z)$, or
   $a - b - (y + x - z)$, or
   $a - b - (-z + x + y)$.

84. When two or more parentheses occur in the same expression, they are removed by the same rule, beginning with the *interior* parenthesis.

Thus, $a-[b-c-(d+x)+e]=a-(b-c-d-x+e)=a-b+c+d+x-e$.

NOTE.—Quantities may be included in more than one parenthesis, by observing the preceding rules.

Remove the parentheses from the following expressions:

3. $ab - (bc - d)$. *Ans.* $ab - bc + d$.
4. $b - (c - d + m)$.
5. $5x - (-y + ab - 4d)$.
6. $2a - [b + c - (x + y) - d]$.
7. $a - (b - c) - (a - c) + c - (a - b)$.

☞ The principles governing the signs in the use and removal of parentheses should be made familiar by practice.

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82. How when the sign $-$ is prefixed? 83. How inclose terms in a parenthesis with $-$ prefixed to it?

CHAPTER IV.

MULTIPLICATION.

85. ***Multiplication*** is finding the *amount* of a quantity taken or added to itself, a given number of times.

Thus, 3 times 4 are 12, and 4 taken 3 times $(4+4+4) = 12$.

The ***Multiplicand*** is the quantity to be multiplied.
The ***Multiplier*** is the quantity by which we multiply.
The ***Product*** is the quantity found by multiplication.

86. The ***Factors*** of a quantity are the multiplier and multiplicand which produce it.

PRINCIPLES.

87. 1°. *The multiplier must be considered an abstract quantity.*

2°. *The product is of the same nature as the multiplicand; for, repeating a quantity does not alter its nature.*

3°. *The product of two or more factors is the same in whatever order they are multiplied.*

CASE I.

88. To Multiply a *Monomial* by a *Monomial*.

1. What is the product of a multiplied by c?

Ans. $a \times c$, or ac.

NOTE.—The product of two or more letters, we have seen, is expressed by writing them one after another, either with or without the sign of multiplication between them. (Art. 10.)

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85. Define multiplication. The multiplicand, Multiplier, Product. 86. Factors. 87. Name Prin. 1. Prin. 2. Prin. 3.

2. If 1 ton of iron costs a dollars, what will x tons cost?

ANALYSIS. x tons will cost x times as much as 1 ton; and x times a dollars are ax dollars. That is, a dollars are taken x times, and are equal to $a+a+a \ldots .$, and so on to x terms.

3. What is the product of $4a$ by $2b$?

ANALYSIS.—Since each coefficient and each letter in the multiplier and multiplicand is a factor, it follows that the answer must be the product of the coefficients with all the letters of both factors annexed. Hence, the

OPERATION.

$$\begin{array}{r} 4a \\ \underline{2b} \\ \textit{Ans. } 8ab \end{array}$$

RULE.—*Multiply the coefficients together, and prefix the product to the product of the literal factors.*

Multiply the following quantities:

4. $4ab$ by $5x$. *Ans.* $20abx$.
5. $6bc$ by $7a$.
6. $7abc$ by $5xy$.
7. $8dm$ by xy.
8. $9bcd$ by $7xyz$.
9. $7xy$ by $8ab$.
10. $6ac$ by $7dx$.
11. $9bd$ by $6cm$.
12. $7xyz$ by $9adf$.

SIGNS OF THE PRODUCT.

89. The investigation of the laws that govern the signs of the product, requires attention to the following

PRINCIPLES.

1°. *Repeating a quantity does not change its sign.*

2°. *The sign of the multiplier shows whether the repetitions of the multiplicand are to be added, or subtracted.*

90. If the ***Signs*** of the factors are *alike*, the sign of the product will be *positive;* if *unlike,* the sign of the product will be *negative.*

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88. How multiply a monomial by a monomial? 89. Name Principle 1. Prin. 2. 90. If signs of factors are alike, what is the sign of the product? If unlike?

91. Demonstration.—There are four points to be proved:

First. That + into + produces +.

Let $+a$ be the multiplicand and $+4$ the multiplier. It is plain that $+a$ taken $+4$ times is $+4a$. (Prin. I.) The sign of the multiplier being +, shows that the product $+4a$, is to be added, which is done by setting it down without changing its sign. (Art. 66.)

Second. That − into + produces −.

Let $-a$ be multiplied by $+4$. Now $-a$ taken 4 times is $-4a$; for a negative quantity repeated is still *negative*. (Prin. I.) But the sign before the multiplier being +, shows that the negative product $-4a$, is to be added. This also is done by setting it down without changing its sign. (Art. 66.)

Third. That + into − produces −.

Let $+a$ be multiplied by -4. We have seen above that $+a$ taken 4 times is $+4a$. But here the sign of the multiplier being −, shows that the product $+4a$, is to be subtracted. This is done by changing its sign from + to −, on setting it down. Thus, $+a \times -4 = -4a$. (Art. 77.)

Fourth. That − into − produces +.

Let $-a$ be multiplied by -4. It has also been shown that $-a$ taken 4 times is $-4a$. But the sign of the multiplier being −, shows that this negative product $-4a$, is to be subtracted. This is also done by changing its sign from − to +, when we set it down. Thus, $-a \times -4 = +4a$. (Art. 77.) Hence, universally,

92. *Factors having like signs produce +, and unlike signs −.*

13. Multiply $+4ab$ by $-7cd$. *Ans.* $-28abcd$.
14. Multiply $-5xy$ by $+9ab$.
15. Multiply $+6ab$ by $+7dc$.
16. Multiply $-8xy$ by $-19abc$.
17. Multiply $+18abc$ by $-23xy$.
18. Multiply $-35xy$ by $-27bcd$.

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91. Prove the first point from the blackboard. The second. Third. Fourth.
92. Rule for signs.

93. When a letter is multiplied into itself, or taken *twice* as a factor, the product is represented by $a \times a$, or aa; when taken *three times*, by aaa, and so on, forming a series of powers. But powers, we have seen, are expressed by writing the letter *once* only, with the index above it, at the right hand. (Art. 31.)

19. What is the product of aaa into aa?

ANALYSIS. $aaa \times aa = aaaaa$, or a^5, *Ans.* Now $aaa = a^3$, and $aa = a^2$; but adding the exponents of a^3 and a^2 we have a^5, the same as before. Hence,

94. *To multiply powers of the same letter together, add their exponents.*

NOTES.—1. All powers of 1 are 1.

2. When a letter has no exponent, 1 is always understood.

Multiply the following quantities:

20. ab^2c^3 by a^2bc. *Ans.* $a^3b^3c^4$.

21. $3a^2b^2x$ by $2ab^2y$. *Ans.* $6a^3b^4xy$.

22. $3xy^2$ by $5x^2$.

23. $6a^2b$ by $4ab^2$.

24. a^4x^2y by a^2x^3y.

25. ab^m by ab^n.

26. $3xyz$ by $2xy$.

27. $6a^2b^2c$ by $3a^3b^2c$.

28. If $a = 3$, what is the difference between $3a$ and a^3?

29. If $x = 4$, what is the difference between $4x$ and x^4?

95. The preceding principles illustrating monomials may be summed up in the following

RULE.—*Multiply the coefficients and letters of both factors together; to the product prefix the proper sign, and give to each letter its proper index.*

NOTE.—It is immaterial in what order the factors are taken, but it is more convenient, and therefore customary, to arrange the letters in alphabetical order. (Art. 87, Prin. 3.)

30. Multiply $-3xy$ by $-2x$.

31. Multiply $6a^2b^2$ by $-3a^2bc$.

---

94. How multiply powers of the same letter together? 95. What is the rule for multiplying monomials?

	(32.)	(33.)	(34.)	(35.)
Multiply	$4xy$	$7ab^2$	$8c^2x^n$	$4a^3b^2$
By	x^2y	$3a^2b$	$5c^2y$	$-7ab$

	(36.)	(37.)	(38.)	(39.)
Multiply	$3xy$	$7abc^2$	$4a^2c^3$	xyz^2
By	$-2xy^2$	$3abc^2$	$-7ac$	xyz

CASE II.

96. To Multiply a *Polynomial* by a *Monomial*.

1. What is the product of $a + b$ multiplied by b?

ANALYSIS.—Multiplying each term of the multiplicand by the multiplier, we have $a \times b = ab$, and $b \times b = b^2$. The result, $ab + b^2$, is the product required. Hence, the

OPERATION.

$$\begin{array}{r} a + b \\ b \\ \hline \textit{Ans.}\ ab + b^2 \end{array}$$

RULE.—*Multiply each term of the multiplicand by the multiplier; giving each partial product its proper sign, and each letter its proper index.*

Multiply the following quantities:

2. $bc - ad$ by ab. *Ans.* $ab^2c - a^2bd$.
3. $3ax^2 + 4cd$ by $2c$.
4. $5ab^2 - 2cd + x$ by $3ax$.
5. $4a^2 - 3ab + m^2$ by $-2bd$.
6. $3a^2 - 4b^2 - 2c^2$ by $-5a^2c$.

CASE III.

97. To Multiply a *Polynomial* by a *Polynomial*.

7. What is the product of $a + b$ into $a + b$?

ANALYSIS.—Since the multiplicand is to be taken as many times as there are units in the multiplier, the product must be equal to a times $a+b$ added to b times $a+b$. Now a times $a+b = a^2+ab$, and b times $a+b = +ab+b^2$. Hence, $a+b$ times $a+b$ must be equal to $a^2+2ab+b^2$.

OPERATION.

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \qquad \\ + ab + b^2 \\ \hline \textit{Ans.}\ a^2 + 2ab + b^2 \end{array}$$

96. How multiply a polynomial by a monomial?

8. Multiply $b + 2a - 3c$ by $a + b$.

Analysis.—We multiply each term in the multiplicand by each term in the multiplier, giving to each product the proper sign. (Art. 89.) Finally, we add the partial products, and the result is the answer required.

OPERATION.

$$\begin{array}{rl} & 2a + b - 3c \\ & a + b \\ \hline & 2a^2 + ab - 3ac \\ & \quad + 2ab \qquad + b^2 - 3bc \\ \hline \textit{Ans.} & 2a^2 + 3ab - 3ac + b^2 - 3bc \end{array}$$

98. The various principles developed in the preceding cases, may be summed up in one

GENERAL RULE.

Multiply each term of the multiplicand by each term of the multiplier, giving each product its proper sign, and each letter its proper exponent.

The sum of the partial products will be the true product.

Note.—For *convenience* in adding the partial products, *like* terms should be placed under each other.

Multiply the following quantities:

1. $2a + b$ by $3x + y$.
2. $3x + 4y$ by $a - b$.
3. $4b - c$ by $3d - a$.
4. $6xy - 2a$ by $b + c$.
5. $3a + 4b - c$ by $x - y$.
6. $5x + 3y + z$ by $a + b$.
7. $7cdx - 3ab$ by $2m - 3n$.
8. $8abc + 4m$ by $3x - 4y$.
9. Multiply $3ab^n$ by $8a^2b$. *Ans.* $24a^3b^{n+1}$.
10. Multiply $-7ax^m$ by $-8a^2x^n$. *Ans.* $56a^3x^{m+n}$.
11. Multiply $3abc^n$ by xyz^m.
12. Multiply acd^m by $11bcd^n$.
13. Multiply $-ax^2$ by $-ax^n$.
14. Multiply $x(a+b)^2$ by $c(a+b)^2$. (Art. 15.)
15. Multiply $c(a-b)^3$ by $5(a-b)^2$.
16. Multiply $a(x+y)^m$ by $bc(x+y)^n$.
17. Multiply $3x(a+b)^3$ by $-(a+b)$.

98. How multiply a polynomial by a polynomial?

99. When the polynomials contain *different powers* of the same letter, the terms should be arranged so that the *first* term shall contain the *highest power* of that letter, the second term the *next highest* power, and so on to the last term. This letter is called the *leading letter.*

(18.)

$$\begin{array}{l} a^2+2ab+b^2 \\ \underline{a\;+b\qquad\qquad} \\ a^3+2a^2b+\;ab^2 \\ \underline{\quad+\;a^2b+2ab^2+b^3} \\ a^3+3a^2b+3ab^2+b^3, \textit{ Ans.} \end{array}$$

(19.)

$$\begin{array}{l} 3a^3b+a^2b \\ \underline{4a^2b-3ab} \\ 12a^5b^2+4a^4b^2 \\ \underline{\qquad-9a^4b^2-3a^3b^2} \\ 12a^5b^2-5a^4b^2-3a^3b^2, \textit{ Ans.} \end{array}$$

20. Multiply $a^2 - ab + b^2$ by $a + b$.
21. Multiply $a^2 - ab + b^2$ by $a^2 + ab + b^2$.
22. Multiply $x^2 + x + 1$ by $x^2 - x + 1$.
23. Multiply $3x^2 - 2xy + 5$ by $x^2 + 2xy - 6$.
24. Multiply $4ax - 2ay$ by $6ax + 3ay$.
25. Multiply $d + bx$ by $d + cx$.

100. The ***Multiplication*** of polynomials may be *indicated* by inclosing each factor in a parenthesis, and writing one after the other.

Thus, $(a+b)(a+b)$ is equivalent to $(a+b)\times(a+b)$.

NOTE.—***Algebraic Expressions*** are said to be *developed* or *expanded*, when the operations indicated by their *signs* and *exponents* are performed.

26. Develop the expression $(a + b)(c + d)$.
Ans. $ac + bc + ad + bd$.
27. Develop $(x + y)(x - y)$.
28. Develop $(a^3 + 1)(a + 1)$.
29. Expand $(x^2 + 2xy + y^2)(x + y)$.
30. Expand $(a^m + b^n)(a^m + b^n)$.
31. Expand $(x + y + z)(x + y + z)$.

99. How arrange different powers of the same letter? 100. How indicate the multiplication of polynomials?

GENERAL PRINCIPLES AND FORMULAS OF MULTIPLICATION.

101. The *square* of the *sum* of two quantities is equal to the square of the first, *plus* twice their product, *plus* the square of the second.

1. Let it be required to multiply $a + b$ into itself.

ANALYSIS.—Each term of the multiplicand being multiplied by each term of the multiplier, we have a times $a+b$ and b times $a + b$, the sum of which is $a^2 + 2ab + b^2$. Hence, the

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \qquad \\ + ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

FORMULA. $(a+b)^2 = a^2 + 2ab + b^2$. *Ans.* $a^2 + 2ab + b^2$

102. The *square* of the *difference* of two quantities is equal to the square of the first, *minus* twice their product, *plus* the square of the second.

2. Let it be required to multiply $a - b$ by $a - b$.

ANALYSIS.—Reasoning as before, the result is the same, except the sign of the middle term, $2ab$, has the sign $-$ instead of $+$. Hence, the

$$\begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \qquad \\ - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array}$$

FORMULA. $(a-b)^2 = a^2 - 2ab + b^2$. *Ans.* $a^2 - 2ab + b^2$

103. The *product* of the *sum* and *difference* of two quantities is equal to *the difference of their squares.*

3. Let it be required to multiply $a + b$ by $a - b$.

ANALYSIS.—This operation is similar to the last two; but the partial products $+ab$ and $-ab$, being equal, balance each other. Hence, the

$$\begin{array}{r} a + b \\ a - b \\ \hline a^2 + ab \qquad \\ - ab - b^2 \\ \hline a^2 \qquad\quad - b^2 \end{array}$$

FORMULA. $(a+b)(a-b) = a^2 - b^2$. *Ans.* $a^2 - b^2$

REMARK.—The product of the *sum* and *difference* of two *squares* is equal to the difference of their *fourth* powers.

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101. To what is the square of the sum of two quantities equal? 102. The square of the difference? 103. The product of the *sum* and *difference* of two quantities? *Remark.* Of two squares?

104. *The product of the sum of two quantities into a third, is equal to the sum of their products.*

4. Let x and y be two quantities, whose sum is to be multiplied by a. Thus,

The product of the sum $(x+y)\times a \qquad = ax+ay$
The sum of the products of $x\times a+y\times a = ax+ay$
And $ax+ay=ax+ay$. Hence, the

FORMULA. $a(x+y)=ax+ay$.

105. *The product of the difference of two quantities into a third, is equal to the difference of their products.*

5. Let x and y be two quantities, whose difference is to be multiplied by a. Thus,

The product of the difference $(x-y)\times a \qquad = ax-ay$
The difference of the products of $x\times a-y\times a = ax-ay$
And $ax-ay=ax-ay$. Hence, the

FORMULA. $a(x-y)=ax-ay$.

REMARK.—The application of the preceding principles is so frequent in algebraic processes, that it is important for the learner to make them very familiar.

Develop the following expressions by the preceding formulas:

1. $(a+1)(a+1)$.
2. $(2a+1)(2a+1)$.
3. $(2a-b)(2a-b)$.
4. $(x+y)(x+y)$.
5. $(x-y)(x-y)$.
6. $(1+x)(1-x)$.
7. $(7y^2-y)(7y^2-y)$.
8. $(4m-3n)(4m+3n)$.
9. $(x^2-y)(x^2+y)$.
10. $(1-7x)(1+7x)$.
11. $(4x-1)(4x-1)$.
12. $(5b+1)(5b+1)$.
13. $(1-x)(1-x)$.
14. $(1+2x)(1+2x)$.
15. $(8b-3a)(8b-3a)$.
16. $(ab+cd)(ab+cd)$.
17. $(3a-2y)(3a+2y)$.
18. $(x^2+y)(x^2-y)$.
19. $(x-y^2)(x-y^2)$.
20. $(2a^2+x)(2a^2-x)$.

---

104. What is the product of the *sum* of two quantities into a third equal to?
105. Of the *difference*?

PROBLEMS.

106. Problems requiring each side of the equation to be multiplied by equal quantities.

1. George has 1 third as many pears as apples, and the number of both is 24. How many has he of each?

ANALYSIS.—If x represents the number of apples, then $\frac{x}{3}$ will represent the number of pears, and $x+\frac{x}{3}$ will equal 24, the number of both. The denominator of x is removed by multiplying each term on both sides of the equation by 3. (Ax. 6.) The result is $3x+x$, or $4x=72$. Hence, $x=18$, the apples, and $18\div3=6$, the pears. Hence,

$$\begin{aligned} \text{Let}\quad x &= \text{No. apples;} \\ \frac{x}{3} &= \text{pears.} \\ x+\frac{x}{3} &= 24 \\ 3x+x &= 72 \\ 4x &= 72 \\ \therefore\quad x &= 18 \text{ apples.} \\ \frac{x}{3} &= 6 \text{ pears.} \end{aligned}$$

107. When a *term* on either side of the equation has a *denominator*, that denominator is removed *by multiplying every term on both sides of the equation by it.* (Ax. 4.)

2. What number is that, 1 seventh of which is 9?

Ans. 63.

3. What number is that, 2 thirds of which are 24?

4. A man being asked how many chickens he had, answered, 3 fourths of them equal 18. How many had he?

5. What number is that, 1 third and 1 fourth of which are 21?

ANALYSIS.—If x represent the number, then will $\frac{x}{3}+\frac{x}{4}=21$, by the conditions. Multiplying each term on both sides by the denominators 3 and 4 separately, we have $4x+3x=252$. (Ax. 4.) Uniting the terms, $7x=252$, and $x=36$, *Ans.*

$$\begin{aligned} \text{Let}\quad x &= \text{No.} \\ \frac{x}{3}+\frac{x}{4} &= 21 \\ 4x+3x &= 252 \\ \therefore\quad x &= 36 \end{aligned}$$

PROOF. $\frac{1}{3}$ of $36=12$, and $\frac{1}{4}$ of $36=9$. Now, $12+9=21$.

---

107. When a quantity on either side of an equation has a denominator, how remove it?

6. What number is that, 2 thirds of which exceed 1 half of it by 8?

7. A general lost 840 men in battle, which equaled 3 sevenths of his army. Of how many men did his army consist?

8. If 3 eighths of a yacht are worth $360, what is the whole worth?

9. If $\frac{4x}{5}$ equals 20, to what is x equal?

10. If $\frac{5x}{4}$ is equal to 20, to what is x equal?

11. If $\frac{3x}{7}$ is equal to 24, to what is x equal?

12. If $\frac{4x}{11}$ is equal to 28, to what is x equal?

13. Henry has 30 peaches, which are 5 sixths the number of his apples. How many apples has he?

14. A farmer has 3 sevenths as many cows as sheep, and his number of cows was 30. How many sheep had he? How many of both?

15. Divide 28 pounds into two parts, such that one may be 3 fourths of the other.

16. A lad having given 1 third of his plums to one schoolmate, and 1 fourth to another, had 10 left. How many had he at first?

17. What number is that 1 third and 1 sixth of which are 21?

18. What number is that 1 fourth of which exceeds 1 sixth by 12?

19. Divide 36 into two parts, such that one may be 2 thirds of the other?

20. One of my apple trees bore 3 sevenths as many apples as the other, and both yielded 21 bushels. How many bushels did each yield?

CHAPTER V.

DIVISION.

108. ***Division*** is finding how many times one quantity is contained in another.

The ***Dividend*** is the quantity to be divided.

The ***Divisor*** is the quantity by which we divide.

The ***Quotient*** is the quantity found by division.

The ***Remainder*** is a part of the dividend left after division.

109. Division is the *reverse* of multiplication, the *dividend* answering to the *product*, the divisor to *one* of the factors, and the *quotient* to the *other*.

PRINCIPLES.

110. 1°. *When the divisor is a quantity of the same kind as the dividend, the quotient is times, or a number.*

2°. *When the divisor is a number, the quotient is a quantity of the same kind as the dividend.*

3°. *The product of the divisor and quotient is equal to the dividend.*

4°. *Cancelling a factor of a quantity, divides the quantity by that factor.*

CASE I.

111. To Divide a *Monomial* by a Monomial.

1. What is the quotient of $abcd$ divided by cd?

ANALYSIS.—The divisor cd is a factor of the dividend; therefore, if we cancel this factor, the other factor ab, will be the quotient. (Prin. 4.)

OPERATION.

$cd\,)\,\overline{abcd}$

Ans. ab.

---

108. Define division. The dividend. Divisor. Quotient. Remainder. 109. Of what is division the reverse? Explain. 110. Name the first principle. The second. Third. Fourth.

2. What is the quotient of $18ab$ divided by $6a$?

ANALYSIS.—Dividing the coefficient of the dividend by that of the divisor, and cancelling the common factor a, we have $18ab \div 6a = 3b$, the quotient required. (Prin. 1.) Hence, the

OPERATION.

$6a\,)\,\underline{18ab}$

Ans. $3b$.

RULE.—*Divide one coefficient by the other, and to the result annex the quotient of the literal parts.*

Divide the following quantities:

(3.)	(4.)	(5.)	(6.)
$2c\,)\,\underline{4abc}$	$4b\,)\,\underline{20bxy}$	$8xy\,)\,\underline{40xy}$	$16b\,)\,\underline{32ab}$

(7.)	(8.)	(9.)
$9m\,)\,\underline{45abm}$	$20mn\,)\,\underline{60bcmn}$	$24xy\,)\,\underline{96mnxy}$

SIGNS OF THE QUOTIENT.

112. The rule for the signs in division is the same as that in multiplication. That is,

If the divisor and dividend have *like signs*, the sign of the *quotient* will be $+$; if *unlike*, the sign of the quotient will be $-$.

Thus, $+a \times +b = +ab$; hence, $+ab \div +b = +a$.
$-a \times +b = -ab$; hence, $-ab \div +b = -a$.
$+a \times -b = -ab$; hence, $+ab \div -b = -a$.
$-a \times -b = +ab$; hence, $-ab \div -b = +a$.

Divide the following quantities:

10. $-32abc$ by $-4ab$. *Ans.* $8c$.

11. $18abx$ by $-3z$. *Ans.* $-\dfrac{6abx}{z}$.

12. $21abc$ by $-7ab$.

13. $-28bcd$ by $-4cd$.

14. $35cdm$ by $7cm$.

15. $48abc$ by $-8ac$.

16. $63bdfx$ by $9bx$.

17. $-72acgm$ by $8cm$.

---

111. How divide a monomial by a monomial? 112. What is the rule for the signs?

113. To Divide *Powers* of the same letter.

18. Let it be required to divide a^5 by a^3.

Analysis.—The term $a^5 = aaaaa$, and $a^3 = aaa$. Rejecting the factors aaa from the dividend, the result aa, or a^2, is the quotient. Subtracting 3, the index of the divisor, from 5, the index of the dividend, leaves 2, the index of the quotient. That is, $a^5 \div a^3 = a^2$. (Arts. 31, 110. Prin. 4.) Hence, the

Rule.—*Subtract the index of the divisor from that of the dividend.*

Divide the following quantities:

19. d^7 by d^3.
20. x^{11} by x^5.
21. ac^9 by ac^4.
22. xyz^{18} by xyz^7.
23. $16ab^2$ by $4ab$.
24. $6xy$ by $3y^2$.

114. The preceding principles may be summed up in the following

Rule.—*Divide the coefficient of the dividend by that of the divisor; to the result annex the quotient of the literal factors, prefixing the proper sign and giving each letter its proper exponent.*

Proof.—*Multiply the divisor and quotient together, as in arithmetic.*

Note.—If the letters of the divisor are not in the dividend, the division is *expressed* by writing the divisor *under* the dividend, in the form of a fraction.

25. What is the quotient of $5x$ divided by $3y$? *Ans.* $\frac{5x}{3y}$.

26. $-24a^2b^2c^2 \div -3ab$.
27. $-36x^2yz^2 \div 6xyz$.
28. $5a^2b^2 \div ab$.
29. $-7x^3y^2 \div -xy$.
30. $a^4b^3c^2 \div a^3b^2c$.
31. $16a^3b^5c^4 \div 8a^2b^4c^3$.
32. $32x^4y^3z^2 \div 4x^2yz$.
33. $96a^3b^2c \div 12ab$.
34. $84d^5x^3y^2 \div 7d^2xy$.
35. $108abx^4 \div 9a^2bx^2$.
36. $132x^4yz^5 \div 11x^2yz^3$.
37. $121m^4n^2x^3 \div 11m^2nx^3$.

---

113. How divide powers of the same letter? 114. Rule for division of monomials? Proof? If the letters of the divisor are not in the dividend, what is done?

CASE II.

115. To Divide a *Polynomial* by a Monomial.

1. Divide $ab + ac + ad$ by a.

OPERATION.

$$a\,)\,\underline{ab + ac + ad}$$
$$Ans.\ b + c + d$$

ANALYSIS.—Since the factor a enters into each term of the dividend, it is plain that each term of the dividend must be divisible by this factor. Hence, the

RULE.—*Divide each term of the dividend by the divisor, and connect the results by their proper signs.*

NOTE.—If a polynomial which contains the same *factor in every term*, be divided by the other quantities connected by their signs, the quotient will be *that factor*.

Divide the following quantities:

2. $6a^3 + 10a^2 - 14a$ by $2a$. *Ans.* $3a^2 + 5a - 7$.
3. $4a^4 - 8a^3 + 12a^2$ by $-2a^2$. *Ans.* $-2a^2 + 4a - 6$.
4. $ab^2 + ac^3 + ad^4$ by a.
5. $15x^2y + 25xy$ by $5xy$.
6. $6abc - 2a + 8ab$ by $2a$.
7. $-16by^3 + 4y^2$ by $-8y$.
8. $14x^2y - 7xy^2$ by $-7xy$.
9. $xy^2 + xz - x$ by x.
10. $35a + 28b - 42$ by -7.
11. $15a^2b - 15a^2$ by $5a$.
12. $16x^2ac + 12acd^2 - 4xa^2c$ by $-4ac$.
13. $4a^4 - 20a^2 + 8ab$ by $4a$.
14. $3ab + 15a^2b - 27a^2bd$ by $3ab$.
15. $8a^2bc - 16ab^2c - 20abc^2$ by $4abc$.
16. $6x(a + b)^2 + 9x^2(a + b)^2$ by $3x$.
17. $15(x - y) + 30(x - y)$ by 5.
18. $ax^2(b - c) - a^2x(b - c)$ by ax.
19. $18a^4(a + b)^2 - 12a^3(a + b)^2$ by $6a^2(a + b)^2$.
20. $a^{n+1} - a^{n+2} + a^{n+3}$ by a^n.

---

115. How divide a polynomial by a monomial?

CASE III.

116. To Divide a *Polynomial* by a Polynomial.

1. Divide $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^2 + 2ab + b^2$.

OPERATION.

$$\begin{array}{l|l} a^3+3a^2b+3ab^2+b^3 & a^2+2ab+b^2 \\ \cline{2-2} a^3+2a^2b+\ \ ab^2 & a+b \text{ Quot.} \\ \cline{1-1} \qquad a^2b+2ab^2+b^3 & \\ \qquad a^2b+2ab^2+b^3 & \\ \cline{1-1} \end{array}$$

ANALYSIS. — For *convenience*, we arrange the terms so that the first or leading letter of the divisor shall be the first letter of the dividend. The powers of this letter should be arranged in order, both in the divisor and dividend, the *highest* power standing *first*, the *next highest next*, and so on. The divisor may be placed on the left of the dividend, or on the right, and the quotient under it, at pleasure.

Proceeding as in arithmetic, we find the first term of the divisor is contained in the first term of the dividend a times. Placing the a in the quotient under the divisor, we multiply the whole divisor by it, subtract the product, and to the remainder bring down as many other terms as necessary to continue the operation. Dividing as before, a^2 is contained in a^2b, $+b$ times. Multiplying the divisor by $+b$ and subtracting the product, the dividend is exhausted; therefore $a+b$ is the quotient. Hence, the

RULE.—I. *Arrange the divisor and dividend according to the powers of one of their letters; and finding how many times the first term of the divisor is contained in the first term of the dividend, place the result in the quotient.*

II. *Multiply the whole divisor by the term placed in the quotient; subtract the product from the dividend, and to the remainder bring down as many terms of the dividend as the case may require.*

Repeat the operation till all the terms of the dividend are divided.

NOTE.—If there is a remainder after all the terms of the dividend are brought down, *place it over the divisor*, and *annex* it to the quotient.

---

116. How divide a polynomial by a polynomial? If there is a remainder, what is done with it?

2. Divide $4a^2 + 4ab + b^2$ by $2a + b$. *Ans.* $2a + b$.
3. Divide $x^2 + 2xy + y^2$ by $x + y$.
4. Divide $a^2 - 2ab + b^2$ by $a - b$.
5. Divide $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.
6. Divide $ac + bc + ad + bd$ by $a + b$.
7. Divide $ax + bx - ad - bd$ by $a + b$.
8. Divide $2x^2 + 7xy + 6y^2$ by $x + 2y$.
9. Divide $a^2 - b^2$ by $a + b$.
10. Divide $x^2 - y^2$ by $x - y$.
11. Divide $a^3 - b^3$ by $a - b$.
12. Divide $6a^2 + 13ab + 6b^2$ by $2a + 3b$.
13. Divide $a^2 - a - 6$ by $a - 3$.
14. Divide $a^3 - 3a^2x + 3ax^2 - x^3$ by $a - x$.
15. Divide $6x^4 - 96$ by $3x - 6$.
16. Divide $x^2 + 7x + 10$ by $x + 2$.
17. Divide $x^2 - 5x + 6$ by $x - 3$.
18. Divide $c^2 - 2cx + x^2$ by $c - x$.
19. Divide $a^2 + 2ab + b^2$ by $a + b$.
20. Divide $22(a - b)^2$ by $11(a - b)$.

PROBLEMS.

1. A father being asked the age of his son, replied, My age is 5 times that of my son, lacking 4 years; and the sum of our ages is 56 years. How old was each?

2. John and Frank have 60 marbles, the former having 3 times as many as the latter. How many has each?

3. The sum of two numbers is 72, one of which is 5 times the other. What are the numbers?

4. A man divided 57 pears between two girls, giving one 4 times as many as the other, lacking 3. How many did each have?

5. Three boys counting their money, found they had 190 cents; the second had twice as many cents as the first, and the third as many as both the others, plus 4 cents. How many cents had each?

6. A farmer has 9 times as many sheep as cows, and the number of both is 200. How many of each?

7. Divide 57 into two such parts that the greater shall be 3 times the less, plus 3. What are the numbers?

8. Given $2x + 4x + x - 3 = 60$, to find x.

9. A and B are 35 miles apart, and travel toward each other, A at the rate of 4 miles an hour, and B, 3 miles. In how many hours will they meet?

10. Given $a + 5a + 6a + 2a + 7 = 119$, to find a.

11. Given $8b + 5b + 7b - 10 = 130$, to find b.

12. A lad having 60 cents, bought an equal number of pears, oranges, and bananas; the pears being 3 cents apiece, the oranges 4 cents, and the bananas 5 cents. How many of each did he buy?

13. A cistern filled with water has two faucets, one of which will empty it in 5 hours, the other in 20 hours. How long will it take both to empty it?

14. Given $x + \frac{x}{3} = 45$, to find x.

15. What number is that, to the half of which if 3 be added, the sum will be 8?

16. Three boys have 42 marbles; B has twice as many as A, and C three times as many as A. How many has each?

17. If A has $2x$ dollars, and B twice as many as A, and C twice as many as B, how many have all?

18. Divide 40 into 3 parts, so that the second shall be 3 times the first, and the third shall be 4 times the first.

19. A man divided 60 peaches among 3 boys, in such a manner that B had twice as many as A, and C as many as A and B. How many did each receive?

20. Divide 48 into 3 such parts, that the second shall be equal to twice the first, and the third to the sum of the first and second?

21. What number is that, to three-fourths of which if 5 be added, the sum will be 23?

CHAPTER VI.

FACTORING.

117. ***Factors*** are quantities which multiplied together produce another quantity. (Art. 86.)

118. A ***Composite Quantity*** is the product of two or more *integral factors,* each of which is *greater* than a unit.

Thus, $3a$, $5b$, also x^2y^2, are composite quantities.

119. ***Factoring*** is resolving a composite quantity into its factors. It is the converse of multiplication.

120. An ***Exact Divisor*** of a quantity is one that will divide it without a remainder. Hence,

NOTE.—The ***Factors*** of a quantity are always *exact divisors* of it, and *vice versa.*

121. A ***Prime Quantity*** is one which has no *integral* divisor, except *itself* and 1.

Thus, 5 and 7, also a and b, are prime quantities. Hence,

NOTE.—The *least divisor* of a composite quantity is a prime factor.

122. Quantities are ***prime to each other*** when they have no common *integral divisor,* except the unit 1.

Thus, 11 and 15, also a and bc, are prime to each other.

123. A ***Multiple*** is a quantity which can be divided by another quantity without a remainder. Hence,

A multiple is a *product* of two or more *factors.*

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117. What are factors? 118. A composite quantity? 119. What is factoring? 120. An exact divisor? 121. A prime quantity? 122. When prime to each other? 123. A multiple?

PRINCIPLES.

124. 1°. *If one quantity is an exact divisor of another, the former is also an exact divisor of any multiple of the latter.*

Thus, 3 is a divisor of 6; it is also a divisor of 3×6, of 5×6, etc.

2°. *If a quantity is an exact divisor of each of two other quantities, it is also an exact divisor of their sum, their difference, or their product.*

Thus, 3 is a divisor of 9 and 15, respectively; it is also a divisor of $9 + 15$, or 24; of $15 - 9$, or 6; and of 15×9, or 135.

3°. *A composite quantity is divisible by each of its prime factors, by the product of two or more of them, and by no other quantity.*

Thus, the prime factors of 30 are 2, 3, and 5. Now 30 is divisible by 2, by 3, and by 2×3; by 2×5; by 3×5; by $2 \times 3 \times 5$, and by no other number.

CASE I.

125. To Find the Prime Factors of Monomials.

1. What are the prime factors of $12a^2b$?

ANALYSIS.—The coefficient $12 = 2 \times 2 \times 3$, and $a^2b = aab$. Therefore the prime factors of $12a^2b$ are $2 \times 2 \times 3aab$. Hence, the

RULE.—*Find the prime factors of the numeral coefficients, and annex to them the given letters, taking each as many times as there are units in its exponent.*

NOTE.—In monomials, each letter is a factor. Hence, the prime factors of *literal* monomials are apparent at sight.

Resolve the following quantities into their prime factors:

2. $15x^2y^3$. *Ans.* $3 \times 5\, x\, x\, y\, y\, y$.
3. $18a^2b^2$.
4. $20bx^3y^2$.
5. $35a^3b^2c^2$.
6. $21xy^2z^3$.
7. $17x^2y^3z$.
8. $25ab^2cx^3$.
9. $77a^2bc^2d$.
10. $65m^2n^3x$.

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124. Name Principle 1. Principle 2. Principle 3. 125. How find the prime factors of monomials?

CASE II.

126. To Factor a Polynomial.

1. Resolve $4a^2b + 8ab - 6ac$ into two factors.

OPERATION.

$$\begin{array}{r} 2a\,)\,\underline{4a^2b + 8ab - 6ac} \\ 2ab + 4b - 3c \end{array}$$

$$Ans.\ 2a\,(2ab + 4b - 3c)$$

ANALYSIS.—By inspection, we perceive the factor $2a$ is common to each term; dividing by it, the quotient $2ab+4b-3c$ is the other factor. For convenience, we enclose this factor in a parenthesis, and prefix to it the factor $2a$, as a coefficient.

PROOF.—The factor $(2ab+4b-3c) \times 2a = 4a^2b+8ab-6ac$. Hence, the

RULE.—*Divide the polynomial by the greatest common monomial factor; the divisor will be one factor, the quotient the other.* (Art. 115.)

NOTE.—Any *common* factor, or the *product* of any two or more common factors, may be taken as a divisor; but the result will vary in form according to the factors employed. (Ex. 2.)

2. Resolve $a^2b + ab^2$ into two factors, one of which shall be a monomial. *Ans.* $ab\,(a+b)$, $a\,(ab+b^2)$, or $b\,(a^2 + ab)$.

3. Factor $a + ab + ac$. *Ans.* $a\,(1 + b + c)$.
4. Factor $by + bc + 3bx$.
5. Factor $2ax + 2ay - 4az$.
6. Factor $3bcx - 6bcx - 3abc$.
7. Factor $8dmn - 24dm$.
8. Factor $35am + 14ax$.
9. Factor $27bdx - 54dmy$.
10. Factor $6a^2b + 9a^2c$.
11. Factor $21ax^2y + 35axy$.
12. Factor $25 + 15x^2 - 20x^2y^2$.
13. Factor $x + x^2 + x^3$.
14. Factor $3x + 6 - 9y$.
15. Factor $19a^5x - 19a^5$.

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126. How factor a polynomial?

CASE III.

127. To Resolve a *Trinomial* into two equal Binomial Factors.

1. Resolve $x^2 + 2xy + y^2$ into two equal binomial factors.

ANALYSIS.—Since the *square* of a quantity is the product of two equal factors (Art. 30), it follows that the *square root* of a quantity is one of the two equal factors which produce it. (Art. 32.) Therefore the square root of x^2 is x, that of y^2 is y. And since the middle term $2xy$ is twice the product of these two terms, $x^2+2xy+y^2$ must be the square of the binomial $x+y$. Consequently, $x+y$ is one of the two equal binomial factors.

OPERATION.

$$\sqrt{x^2} = x, \quad \sqrt{y^2} = y,$$
$$\therefore \quad x^2 + 2xy + y^2 =$$
$$(x+y)(x+y), \ Ans.$$

2. Resolve $x^2 - 2xy + y^2$ into two equal binomial factors.

ANALYSIS.—Reasoning as before, the quantity $x^2-2xy+y^2$ is the square of the residual $x-y$. Therefore, the two equal factors must be $x-y$ and $x-y$. Hence, the

OPERATION.

$$\sqrt{x^2} = x, \quad \sqrt{y^2} = y$$
$$\therefore \quad x^2 - 2xy + y^2 =$$
$$(x-y)(x-y), \ Ans.$$

RULE.—*Find the square root of each of the square terms, and connect these roots by the sign of the middle term.*

NOTE.—A *trinomial*, in order to be resolved into equal *binomial* factors, must have two of its terms squares, and the other term *twice the product* of their square roots. (Art. 101.)

Resolve the following into two equal binomials:

3. $a^2 + 2ab + b^2$.
4. $x^2 - 2xy + y^2$.
5. $m^2 + 4mn + 4n^2$.
6. $16a^2 + 8a + 1$.
7. $49 + 70 + 25$.
8. $4a^2 - 12ab + 9b^2$.
9. $y^2 + 2y + 1$.
10. $1 - 2c^2 + c^4$.
11. $x^{2m} + 2x^my^n + y^{2n}$.
12. $4a^{2n} - 4a^n + 1$.
13. $a^4 + 2a^2b^2 + b^4$.
14. $a^2x^4 + 2ax^2y + y^2$.

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127. How resolve a trinomial into equal binomial factors?

CASE IV.

128. To Factor a Binomial consisting of the *Difference* of two Squares.

1. Resolve $4a^2 - 9b^2$ into two binomial factors.

OPERATION.

$$\sqrt{4a^2} = 2a$$
$$\sqrt{9b^2} = 3b$$
$$\therefore\ 4a^2 - 9b^2 = (2a+3b)(2a-3b),\ Ans.$$

ANALYSIS.—Both of these terms are squares; the root of the first is $2a$, that of the second is $3b$. But the *difference* of the squares of two quantities is equal to the *product* of their sum and difference. (Art. 103.) Now the sum of these two quantities is $2a + 3b$, and the difference is $2a - 3b$; therefore, $4a^2 - 9b^2 = (2a+3b)(2a-3b)$. Hence, the

RULE.—*Find the square root of each term. The sum of these roots will be one factor, and their difference the other.*

NOTE.—This rule is one of the numerous applications of the formula contained in Art. 103.

2. Resolve $a^2 - x^2$ into two binomial factors.
3. Resolve $9x^2 - 16y^2$ into two binomial factors.
4. Resolve $y^2 - 4$ into two binomial factors.
5. Resolve $9 - x^2$ into two binomial factors.
6. Resolve $a^2 - 1$ into two binomial factors.
7. Resolve $1 - b^2$ into two binomial factors.
8. Resolve $25a^2 - 16b^2$ into two binomial factors.
9. Resolve $4x^2 - y^2$ into two binomial factors.
10. Resolve $1 - 16a^2$ into two binomial factors.
11. Resolve $25 - 1$ into two binomial factors.
12. Resolve $x^4 - y^4$ into two binomial factors.
13. Resolve $a^2x^2 - b^2y^2$ into two binomial factors.
14. Resolve $m^4 - n^4$ into two binomial factors.
15. Resolve $a^{2m} - b^{2n}$ into two binomial factors.

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128. How factor a binomial consisting of the difference of two squares?

CASE V.

129. Various classes of examples of *higher powers* may be factored by means of the following

PRINCIPLES.

1°. *The difference of any two powers of the same degree is divisible by the difference of their roots.*

Thus, $(x^2-y^2) \div (x-y) = x+y.$

$(x^3-y^3) \div (x-y) = x^2+xy+y^2.$

$(x^4-y^4) \div (x-y) = x^3+x^2y+xy^2+y^3.$

$(x^5-y^5) \div (x-y) = x^4+x^3y+x^2y^2+xy^3+y^4.$

2°. *The difference of two even powers of the same degree is divisible by the sum of their roots.*

Thus, $(x^2-y^2) \div (x+y) = x-y.$

$(x^4-y^4) \div (x+y) = x^3-x^2y+xy^2-y^3.$

$(x^6-y^6) \div (x+y) = x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5.$

3°. *The sum of two odd powers of the same degree is divisible by the sum of their roots.*

Thus, $(x^3+y^3) \div (x+y) = x^2-xy+y^2.$

$(x^5+y^5) \div (x+y) = x^4-x^3y+x^2y^2-xy^3+y^4.$

$(x^7+y^7) \div (x+y) = x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6$, etc.

NOTE.—The *indices* and *signs* of the quotient follow regular laws:

1st. The *index* of the *first letter* regularly *decreases* by 1, while that of the *following letter increases* by 1.

2d. When the *difference* of *two powers* is divided by the difference of their roots, the *signs* of all the terms in the quotient are *plus*. When their *sum* or *difference* is divided by the *sum* of their roots, the *odd* terms of the quotient are *plus*, and the *even* terms *minus*.

☞ If the principles and examples of this Case are deemed too difficult for beginners, they may be deferred until the Binomial Theorem is explained. (Arts. 268–270.)

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129. Recite Prin. 1. Prin. 2. Prin. 3. *Note.* What is the index of the first letter? Of the following letter? What is said of the signs?

130. To Factor the *Difference* of *any two* Powers of the same Degree.

1. Resolve $x^3 - y^3$ into two factors.

SOLUTION.—The binomial $(x^3-y^3) \div (x-y) = x^2+xy+y^2$. $\therefore x-y$ and x^2+xy+y^2 are the factors. (Prin. 1.) Hence, the

RULE.—*Divide the difference of the powers by the difference of the roots; the divisor will be one factor, the quotient the other.*

Resolve the following into two factors:

2. $x^3 - 1$.
3. $x^6 - y^6$.
4. $x^2 - 1$.
5. $1 - 36y^2$.

131. To Factor the *Difference* of two *even* Powers of the same Degree.

6. Resolve $a^4 - b^4$ into two factors.

SOLUTION.—By Prin. 2, a^4-b^4 is divisible by $a+b$. Thus, $(a^4-b^4) \div (a+b) = a^3-a^2b+ab^2-b^3$, the divisor being one factor, the quotient the other. Hence, the

RULE.—*Divide the difference of the given powers by the sum of their roots; the divisor will be one factor, the quotient the other.* (Art. 129, Prin. 2.)

Resolve the following quantities into two factors:

7. $b^2 - x^2$.
8. $d^4 - z^4$.
9. $a^6 - b^6$.
10. $x^4 - 1$.
11. $1 - a^6$.
12. $a^8 - 1$.

132. To Factor the *Sum* of two *odd* Powers of the same Degree.

13. Resolve $a^3 + b^3$ into two factors.

SOLUTION.—Dividing a^3+b^3 by $a+b$, the factors are $a+b$ and a^2-ab+b^2. (Prin. 3.) Hence, the

RULE.—*Divide the sum of the powers by the sum of the roots; the divisor and quotient are the factors.*

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130. How factor the difference of any two powers of the same degree? 131. How factor the difference of two even powers of the same degree? 132. The sum of two odd powers of the same degree?

Resolve the following quantities into two factors:

14. $x^5 + y^5$.
15. $a^3 + 1$.
16. $a^5 + 1$.
17. $1 + y^3$.
18. $1 + a^5$.
19. $1 + b^7$.

133. It will be observed that in the preceding examples of this Case, binomials have been resolved into *two factors*. These factors may or may not be *prime* factors.

Thus, in Ex. 6, $a^4 - b^4 = (a+b)(a^3 - a^2b + ab^2 - b^3)$. But the factor $(a^3 - a^2b + ab^2 - b^3)$ is a composite quantity $= (a-b)(a^2+b^2)$.

134. When a binomial is to be resolved into *prime* factors, it should first be resolved into two factors, one of which is *prime;* then the composite factor should be treated in like manner.

20. Let it be required to find the prime factors of $a^4 - b^4$.

SOLUTION.—The $\sqrt{a^4} = a^2$, and $\sqrt{b^4} = b^2$. (Art. 128.) Now $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$. But $a^2 - b^2 = (a+b)(a-b)$. (Art. 103.) Therefore the prime factors of $a^4 - b^4$ are $(a^2+b^2)(a+b)(a-b)$.

Resolve the following quantities into their prime factors:

21. $a^4 - 1$. *Ans.* $(a^2 + 1)(a + 1)(a - 1)$.

22. $1 - y^4$. *Ans.* $(1 + y^2)(1 + y)(1 - y)$.

23. $x^6 - y^6$.
Ans. $(x^2 - xy + y^2)(x^2 + xy + y^2)(x + y)(x - y)$.

24. $x^4 - 2x^2y^2 + y^4$.
Ans. $(x^2 - y^2)^2 = (x + y)(x + y)(x - y)(x - y)$.

25. $x^6 - 1$.
Ans. $(x + 1)(x - 1)(x^2 + x + 1)(x^2 - x + 1)$.

26. $a^6 + 2a^3b^3 + b^6$.
Ans. $(a + b)(a + b)(a^2 - ab + b^2)(a^2 - ab + b^2)$.

27. $a^2 + 9a + 18$. *Ans.* $(a + 6)(a + 3)$.

28. $4a^2 - 12ab + 9b^2$. *Ans.* $(2a - 3b)(2a - 3b)$.

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134. How resolve a binomial into *prime* factors?

CHAPTER VII.

DIVISORS AND MULTIPLES.

135. A ***Common Divisor*** is one that will divide two or more quantities without a remainder.

136. ***Commensurable Quantities*** are those which have a *common* divisor.

Thus, ab^2 and abc are commensurable by ab.

137. ***Incommensurable Quantities*** are those which have no common divisor. (Art. 122.)

Thus, ab and xyz are incommensurable.

138. To Find a Common Divisor of two or more Quantities.

1. Find a common divisor of abx, acy, and adz.

ANALYSIS.—Resolving the given quantities into factors, we perceive the factor a, is common to each quantity, and is therefore a common divisor of them. (Art. 119.) Hence, the

OPERATION.

$$abx = a \times b \times x$$
$$acy = a \times c \times y$$
$$adz = a \times d \times z$$

Ans. a.

RULE.—*Resolve each of the given quantities into factors, one of which is common to all.*

Find a common divisor of the following quantities:

2. $3abcd$ and $9abm$. *Ans.* $3ab$.
3. x^2yz and $2abx$.
4. a^2b, bcd, ab^2xy.
5. $2abc$, acx^2, a^3cy.
6. $2ax$, $6bx$, $14cx$.
7. $35mn$, $7m^2$, $42m^3x$.
8. $24a^2b$, $12ab^2$, $6a^3b^2$.

---

135. What is a common divisor? 136. Commensurable quantities? 137. Incommensurable quantities? 138. How find a common divisor of two or more quantities?

139. The ***Greatest Common Divisor*** of two or more quantities is the greatest quantity that will divide each of them without a remainder.

NOTES.—1. A *common* divisor of two or more quantities is always a *common factor* of those quantities, and the *g. c. d. is their *greatest common factor*.**

2. A *common* divisor is often called a *common measure*, and the *greatest common* divisor, the *greatest common measure*.

PRINCIPLES.

140. 1°. *The greatest common divisor of two or more quantities is the product of all their common prime factors.*

2°. *A common divisor of two quantities is not altered by multiplying or dividing either of them by any factor not found in the other.*

Thus, 3 is a common divisor of 18 and 6; it is also a common divisor of 18, and of (6×5) or 30.

3°. *Changing the signs of a polynomial is the same in effect as dividing it by* -1.

Thus, $(-3a+4b-5c) \div -1 = 3a-4b+5c$. (Art. 112.) Hence,

4°. *The signs of the divisor, or of the dividend, or of both, may be changed without changing the common divisor.*

141. To Find the Greatest Common Divisor of Monomials by *Prime Factors*.

1. What is the ***g. c. d.*** of $35acx$, $28abc$, and $21ay$?

ANALYSIS. — Resolving the given quantities into their prime factors, 7 and a only are common to each; therefore their product $7 \times a$, is the ***g. c. d.*** required. (Prin. 1.)

OPERATION.

$$35acx = 5 \times 7 \times a \times c \times x$$
$$28abc = 2 \times 2 \times 7 \times a \times b \times c$$
$$21ay = 3 \times 7 \times a \times y$$
$$\therefore \quad 7 \times a = 7a. \quad \textit{Ans.}$$

139. What is the greatest common divisor of two or more quantities? *Note* 1. What is true of a common divisor of two or more quantities? Of the *g. c. d.*? 140. Name Principle 1. Principle 2. Principle 3.

* The initials ***g. c. d.*** are used for the greatest common divisor.

2. Find the *g. c. d.* of $4a^3b^2c$, $10a^2b^3$, and $14abdx$.

OPERATION.

$$4a^3b^2c = 2 \times 2 \times aaabbc$$
$$10a^2b^3 = 2 \times 5 \times aabbb$$
$$14abdx = 2 \times 7 \times abdx$$
$$Ans. \quad 2 \times a \times b = 2ab$$

ANALYSIS.—Resolving these quantities into their prime factors, the factor 2 is common to the coefficients; also, a and b are common to the literal parts. Now multiplying these common factors together, we have $2 \times a \times b = 2ab$, which is the *g. c. d.* required. (Prin. 1.) Hence, the

RULE.—*Resolve the given quantities into their prime factors; and the product of the factors common to all, will be the greatest common divisor.* (Prin. 1.)

NOTE.—In finding the *common prime factors* of the literal part, give each letter the *least* exponent it has in either of the quantities.

3. Find the *g. c. d.* of $6a^2c^2$ and $9abc$.
4. Of $16a^2xy$ and $18acx^2y$.
5. Of $12a^3b^2x^5z^3$ and $16a^5x^2z^3$.
6. Of $6ab^2x^4z^5$, $12a^5x^2z^2$, and $18a^5x^3z^3$.

142. To Find the Greatest Common Divisor of Quantities by *Continued Division*.

1. Required the greatest common divisor of $30x$ and $42x$.

OPERATION.

$$\begin{array}{l} 30x\,)\,42x\,(\,1 \\ \quad\;\; \underline{30x} \\ \quad\;\; 12x\,)\,30x\,(\,2 \\ \qquad\qquad \underline{24x} \\ \qquad\qquad 6x\,)\,12x\,(\,2 \\ \qquad\qquad\qquad \underline{12x} \end{array}$$

ANALYSIS.—If we divide the *greater* quantity by the *less*, the quotient is 1, and $12x$ remainder. Next, dividing the first divisor $30x$, by the first remainder $12x$, the quotient is 2 and the remainder $6x$. Again, dividing the second divisor by the second remainder, the quotient is 2 and no remainder. The last divisor, $6x$, is the *g. c. d.*

DEMONSTRATION.—Two points are required to be proved:

1st. That $6x$ is a *common* divisor of the given quantities.

2d. That $6x$ is their *greatest* common divisor.

First. We are to prove that $6x$ is a *common* divisor of $30x$ and $42x$. By the last division, $6x$ is contained in $12x$, 2 times. Now as $6x$ is a

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141. How find the *g. c. d.* of monomials by prime factors? *Note.* In finding the prime factors of the literal part, what exponents are given?

divisor of $12x$, it is also a divisor of the *product* of $12x$ into 2, or $24x$. (Art. 124, Prin. 1.) Next, since $6x$ is a divisor of itself and $24x$, it must be a divisor of the *sum* of $6x+24x$, or $30x$, which is the *smaller* quantity. For the same reason, since $6x$ is a divisor of $12x$ and $30x$, it must also be a divisor of the *sum* of $12x+30x$, or $42x$, which is the *larger* quantity. Hence, $6x$ is a *common* divisor of $30x$ and $42x$.

Second. We are to prove that $6x$ is the *greatest* common divisor of $30x$ and $42x$.

If the greatest common divisor is not $6x$, it must be either *greater* or *less* than $6x$. But we have shown that $6x$ is a *common* divisor of the given quantities; therefore, no quantity *less* than $6x$ can be the *greatest* common divisor of them. The assumed quantity must therefore be *greater* than $6x$. By supposition, this assumed quantity is a divisor of $30x$ and $42x$; hence, it must be a divisor of their *difference*, $42x-30x$, or $12x$. And as it is a divisor of $12x$, it must also divide the *product* of $12x$ into 2, or $24x$.

Again, since the assumed quantity is a divisor of $30x$ and $24x$, it must also be a divisor of their *difference*, which is $6x$; that is, a *greater* quantity will divide a *less* without a *remainder*, which is impossible. Therefore, $6x$ must be the *greatest common divisor* of $30x$ and $42x$, the second point to be proved. Hence, the

RULE.—*Divide the greater quantity by the less, then divide the first divisor by the first remainder, the second divisor by the second remainder, and so on, till there is no remainder. The last divisor will be the greatest common divisor.*

NOTES.—1. If there are more than two quantities, find the *g. c. d.* of the smaller two, then of this common divisor and a third quantity, and so on with all the quantities.

2. The ***g. c. d. of Polynomials*** is found by the same rule, and may be demonstrated in the same manner.

2. What is the ***g. c. d.*** of $48a$, $72a$, and $108a$.

SUGGESTION.—The greatest common divisor of $48a$ and $72a$ is $24a$; and that of $24a$ and $108a$ is $12a$. Therefore, $12a$ is the greatest common divisor required.

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142. How find the *g. c. d.* of polynomials? Show upon the blackboard the truth of this rule?

3. What is the ***g. c. d.*** of $4a^3 - 21a^2 + 15a + 20$ and $a^2 - 6a + 8$?

OPERATION.

$$\begin{array}{l|ll}
4a^3 - 21a^2 + 15a + 20 & a^2 - 6a + 8 & \text{1st divisor.} \\
\underline{4a^3 - 24a^2 + 32a} & 4a + 3 & \text{1st quotient.} \\
\quad + 3a^2 - 17a + 20 & & \\
\quad \underline{+ 3a^2 - 18a + 24} & & \\
\end{array}$$

$$\begin{array}{l|ll}
a^2 - 6a + 8 & a - 4 & \text{1st remainder and 2d divisor.} \\
\underline{a^2 - 4a} & a - 2 & \text{2d quotient.} \\
\quad - 2a + 8 & & \\
\quad \underline{- 2a + 8} & & \textit{Ans. } a - 4.
\end{array}$$

ANALYSIS.—Dividing the greater quantity by the less, the remainder is $a-4$. Again, dividing the first divisor by the first remainder, the quotient is $a-2$, and no remainder. The last divisor, $a-4$, is the greatest common divisor.

143. It is sometimes necessary, in order to avoid fractions, to *introduce* a factor into one or both the given quantities, or to *cancel* one before finding the greatest common divisor.

It is also sometimes necessary to change the *signs* of the divisor or dividend, or of both. (Art. 140, Prin. 4.)

4. What is the ***g. c. d.*** of $x^2 - 2xy + y^2$ and $x^2 - y^2$?

OPERATION.

$$\begin{array}{rr|ll}
 & x^2 - 2xy + y^2 & x^2 - y^2 & \text{Divisor.} \\
 & \underline{x^2 \qquad - y^2} & 1 & \text{Quotient.} \\
2y\,) & \underline{-2xy + 2y^2} & & \\
\text{Divisor,} & -x + y & & \\
\text{or,} & \underline{x - y} & x^2 - y^2 & \\
\text{Quotient,} & x + y & \underline{x^2 - y^2} &
\end{array}$$

ANALYSIS.—Dividing the greater by the less, the first remainder is $-2xy + 2y^2$. Cancelling from it the common factor $2y$, we have for the second divisor $-x + y$. Changing the signs, it becomes $x-y$. (Art. 140, Prin. 4.) Dividing as before, the quotient is $x+y$, and no remainder. The last divisor, $x-y$, is the greatest common divisor.

---

143. How does it affect the g. c. d. if a factor is introduced into either or both the given quantities? How if one is cancelled? What is true of the signs?

5. What is the *g. c. d.* of $4x^3 - 6x^2 - 4x + 3$ and $2x^3 + x^2 + x - 1$?

OPERATION.

1st dividend,	$4x^3 - 6x^2 - 4x + 3$	$2x^3 + x^2 + x - 1$	1st divisor.
	$4x^3 + 2x^2 + 2x - 2$	2	1st quotient.
2d divisor,	$-8x^2 - 6x + 5$	$8x^3 + 4x^2 + 4x - 4$	2d dividend
2d quotient,	$-x$	$8x^3 + 6x^2 - 5x$	
3d dividend,	$-8x^2 - 6x + 5$	$-2x^2 + 9x - 4$	3d divisor.
	$-8x^2 + 36x - 16$	4	3d quotient.
	$-21\,)\,-42x + 21$		
4th divisor,	$2x - 1$	$-2x^2 + 9x - 4$	4th dividend.
4th quotient,	$-x + 4$	$-2x^2 + x$	
		$8x - 4$	
		$8x - 4$	

$\therefore$ $2x - 1$ is the *g. c. d.*

ANALYSIS.—Dividing the greater by the less, the first term of the first remainder, $-8x^2$, is not contained in $2x^3$, the first term of the second dividend. We therefore multiply this dividend by 4, and it becomes $8x^3+4x^2+4x-4$, and dividing this by the second divisor, the second remainder is $-2x^2+9x-4$. Dividing the preceding divisor by this remainder, we see that the third remainder, $-42x+21$, is not contained in the next dividend. Cancelling the factor -21, the fourth divisor becomes $2x-1$, the greatest common divisor required.

Find the *g. c. d.* of the following quantities:

6. $x^3 - y^3$ and $x^2 - 2xy + y^2$.
7. $a^3 + b^3$ and $a^2 + 2ab + b^2$.
8. $b^2 - 4$ and $b^2 + 4b + 4$.
9. $x^2 - 9$ and $x^2 + 6x + 9$.
10. $a^2 - 3a + 2$ and $a^2 - a - 2$.
11. $a^3 + 3a^2 + 4a + 12$ and $a^3 + 4a^2 + 4a + 3$.
12. $x^3 + 1$ and $x^3 + mx^2 + mx + 1$.
13. $a^5 - b^5$ and $a^2 - b^2$.
14. $a^2 - 5ab + 4b^2$ and $a^3 - a^2b + 3ab^2 - 3b^3$.
15. $3x^5 - 10x^3 + 15x + 8$ and $x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6$.

MULTIPLES.

144. A ***Multiple*** is a quantity which can be divided by another quantity without a remainder. (Art. 123.)

145. A ***Common Multiple*** is a quantity which can be divided by two or more quantities without a remainder.

Thus, $18a$ is a common multiple of 2, 3, 6, and 9.

146. The ***Least Common Multiple*** of two or more quantities is the *least* quantity that can be divided by each of them without a remainder.

Thus, 21 is the least common multiple of 3 and 7; 30 is the least common multiple of 2, 3, and 5.

PRINCIPLES.

147. 1°. *A multiple of a quantity must contain all the prime factors of that quantity.*

Thus, 18 is a multiple of 6, and contains the prime factors of 6, which are 2 and 3.

2°. *A common multiple of two or more quantities must contain all the prime factors of each of the given quantities.*

Thus, 42, a common multiple of 14 and 21, contains all the prime factors of those quantities; viz., 2, 3, and 7.

3°. *The least common multiple of two or more quantities is the least quantity which contains all their prime factors, each factor being taken the greatest number of times it occurs in either of the given quantities.*

Thus, 30 is the least common multiple of 6 and 10, and contains all the prime factors of these quantities; viz., 2, 3, and 5.

---

144. What is a multiple? 145. A common multiple? 146. The least common multiple? 147. Name Principle 1. Principle 2. Principle 3.

148. To Find the Least Common Multiple of Monomials by *Prime Factors.*

1. Find the *l. c. m.** of $15a^4x^2$, $5b^2cx$, and $9bc^3z$.

OPERATION.

$$15a^4x^2 = 3\times 5\times a^4\times x^2$$
$$5b^2cx = 5\times b^2\times c\times x$$
$$9bc^3z = 3\times 3\times b\times c^3\times z$$
Ans. $45a^4b^2c^3x^2z$.

ANALYSIS.—The prime factors of the coefficients are 5, 3, and 3. The prime factors of the letter a are a, a, a, a, which are denoted by a^4. In like manner, the prime factors of x are denoted by x^2, those of b by b^2, and those of c by c^3; z is prime. Taking each of these factors the greatest number of times it occurs in either of the given quantities, the product, $45a^4b^2c^3x^2z$, is the *l. c. m.* required. (Art. 147, Prin. 2.) Hence, the

RULE.—*Resolve the quantities into their prime factors; multiply these factors together, taking each the greatest number of times it occurs in either of the given quantities. The product is the* ***l. c. m.*** *required.*

Or, *Find the least common multiple of the coefficients, and annex to it all the letters, giving each letter the exponent of its highest power in either of the quantities.*

NOTE.—In finding the *l. c. m.* of algebraic quantities, it is often more expeditious to arrange them in a horizontal line, then divide, etc., as in arithmetic.

Required the *l. c. m.* of the following quantities:

2. $9a^3$, $12a^2x^3$, and $24ax^2y$. *Ans.* $72a^3x^3y$.
3. $7ab^2c$, $28bc^3$, and $56a^4b^2d$.
4. $16x^2y^3z$, $20y^4z$, and $8xyz^5$.
5. $15a^3b^2c$, $9ab^4c^2$, and $18a^2bc^5$.
6. $28ab^2$, $14a^2b^4$, $35a^3b^2$, and $42a^4b$.
7. $21x^4y^3z^2$, $35x^3y^2z^5$, and $63xy^3z$.
8. $7m^2n^2y$, $12m^3ny^2$, and $3mn^2y^4$.

---

148. How find the *l. c. m.* of monomials by prime factors? What other method?

* The initials *l. c. m.* are used for the least common multiple.

149. To Find the Least Common Multiple of Polynomials.

9. Required the ***l. c. m.*** of $a^3 + b^3$ and $a^2 - b^2$.

OPERATION.

$$a^2-b^2 = (a+b) \times (a-b)$$
$$a^3+b^3 = (a+b) \times (a^2-ab+b^2)$$
$$(a+b) \times (a-b) \times (a^2-ab+b^2) =$$
$$a^4 - a^3b + ab^3 - b^4. \quad \textit{Ans.}$$

ANALYSIS. — Resolving the quantities into their prime factors, as in the margin, $(a+b)$ is common to both, and is their ***g. c. d.*** (Art. 139.) Now multiplying these factors together, taking each the greatest number of times it occurs in either of the given quantities, the product $a^4-a^3b+ab^3-b^4$ is the ***l. c. m.*** required. (Art. 148.)

SECOND METHOD.

Since the ***g. c. d.*** contains all the factors common to both quantities (Art. 147, Prin. 2), it follows if one of them is divided by the ***g. c. d.*** and the quotient multiplied by the other, the product will be the ***l. c. m.*** Hence, the

RULE.—*Resolve the quantities into their prime factors and multiply these factors together, taking each the greatest number of times it occurs in either of the given quantities. Their product is the* ***l. c. m.*** *required.*

Or, *Find the greatest common divisor of the given quantities, and divide one of them by it. The quotient, multiplied by the other, will be their* ***l. c. m.***

10. Find the ***l. c. m.*** of $2a - 1$ and $4a^2 - 1$.

SOLUTION.—The ***g. c. d.*** is $2a-1$. Now $(4a^2-1) \div (2a-1) = (2a+1)$; and $(2a+1) \times (2a-1) = 4a^2-1$, *Ans.*

Find the ***l. c. m.*** of the following quantities:

11. $x^2 - y^2$ and $x^2 - 2xy + y^2$. *Ans.* $x^3 - x^2y - xy^2 + y^3$.

12. $a^2 - b^2$ and $a^3 - b^3$.

13. $x^2 - 1$ and $x^2 + 2x + 1$.

14. $2a^2 + 3a - 2$ and $6a^2 - a - 1$.

15. $m^2 + m - 2$ and $m^3 - 1$.

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149. How find the *l. c. m.* of polynomials? What other method?

CHAPTER VIII.

FRACTIONS.

150. A ***Fraction*** is one or more of the *equal parts* into which a *unit* is divided.

151. Fractions are expressed by *two quantities* called the *numerator* and *denominator*, one of which is written below the other, with a short line between them.

152. The ***Denominator*** is the quantity *below* the line, and shows into *how many* equal parts the unit is divided.

153. The ***Numerator*** is the quantity *above* the line, and shows *how many* parts are taken.

Thus, the expression $\frac{a}{b}$ shows that the quantity is divided into b equal parts, and that a of those parts are taken.

154. The ***Unit*** or ***Base*** of a fraction is the *quantity* divided into equal parts.

155. The ***Terms*** of a fraction are the *numerator* and *denominator*.

156. An ***Integer*** is a quantity which consists of one or more entire units *only;* as a, $3a$, 5, 7.

157. A ***Mixed Quantity*** is one which contains an integer and a fraction.

Thus, $a + \frac{x}{c}$ is a mixed quantity.

---

150. What is a fraction? 151. How expressed? 152. What does the denominator show? 153. The numerator? 154. What is the base of a fraction? 155. The terms of a fraction? 156. An integer? 157. A mixed quantity?

158. Fractions arise from *division*, the numerator being the dividend and the denominator the divisor. Hence,

159. The ***Value*** of a fraction is the *quotient* of the numerator divided by the denominator.

Thus, the value of 6 thirds is $6 \div 3$, or 2 thirds; of $\frac{12m}{4}$ is $3m$.

SIGNS OF FRACTIONS.

160. ***Every Fraction*** has the sign + or —, expressed or understood, before the dividing line.

161. The ***Dividing Line*** has the force of a *vinculum* or *parenthesis*, and the *sign* before it shows that the *value of the whole fraction* is to be added or subtracted.

162. ***Every Numerator and Denominator*** is preceded by the sign + or —, expressed or understood. In this case, the sign affects only the *single term* to which it is prefixed.

163. If the ***Sign before the Dividing Line*** is changed from + to —, or from — to +, the *value* of the fraction is changed from *positive* to *negative*, or from *negative* to *positive*.

Thus, $a + \frac{bx}{x} = a + b$, but $a - \frac{bx}{x} = a - b$.

164. If ***all the Signs of the Numerator*** are changed, the *value* of the fraction is changed in a corresponding manner.

Thus, $\frac{+ax}{x} = +a$, and $\frac{-ax}{x} = -a$.

---

158. From what do fractions arise? 159. What is the value of a fraction? 160. What is prefixed to the dividing line of a fraction? 161. What is the force of the dividing line? 162. By what is the numerator and denominator preceded? How far does the force of this sign extend? 163. If the sign before the dividing line is changed, what is the effect? 164. If all the signs of the numerator are changed?

165. If ***all the Signs of the Denominator*** are changed, the *value* is also changed in a corresponding manner.

Thus, $\frac{ax}{+x} = +a$; but $\frac{ax}{-x} = -a$. Hence

166. If ***any two*** of these changes are made at the *same* time, they *will balance* each other, and the value of the fraction will not be *altered.*

Thus, $\frac{ax}{x} = +a$. Changing the signs of both numerator and denominator, $\frac{-ax}{-x} = +a$.

PRINCIPLES.

167. The principles for the treatment of fractions in Algebra are the same as those in Arithmetic.

1°. *Multiplying the numerator, or Dividing the denominator,* } *Multiplies the fraction.*

Thus, $\frac{2\times 2}{6} = \frac{4}{6} = \frac{2}{3}$. And $\frac{2}{6\div 2} = \frac{2}{3}$.

2°. *Dividing the numerator, or Multiplying the denominator,* } *Divides the fraction.*

Thus, $\frac{2\div 2}{6} = \frac{1}{6}$. And $\frac{2}{6\times 2} = \frac{2}{12} = \frac{1}{6}$.

3°. *Multiplying, or dividing both terms by the same quantity* } *Does not change its value.*

Thus, $\frac{2\times 2}{6\times 2} = \frac{4}{12} = \frac{2}{6} = \frac{1}{3}$. And $\frac{2\div 2}{6\div 2} = \frac{1}{3}$

4°. *Multiplying and dividing a fraction by the same quantity* } *Does not change its value.*

Thus, $\frac{2\times 2\div 2}{6\times 2\div 2} = \frac{2}{6}$.

165. If all the signs of the denominator are changed? 166. If both are changed? 167. Name Principle 1. Principle 2. Principle 3. Principle 4.

REDUCTION OF FRACTIONS.

168. ***Reduction of Fractions*** is changing their *terms* without altering the *value* of the fractions.

CASE I.

169. To Reduce a Fraction to its *Lowest* Terms.

DEF.—The ***Lowest Terms*** of a fraction are the *smallest terms* in which its numerator and denominator can be expressed. (Art. 122.)

1. Reduce $\dfrac{5a^2b^2dx}{15abcx}$ to its lowest terms.

OPERATION.

$$\frac{5a^2b^2dx}{15abcx} = \frac{abd}{3c}.$$

ANALYSIS.—By inspection, we perceive the factors 5, a, b, and x are common to both terms. Cancelling these common factors, the fraction becomes $\frac{abd}{3c}$. Now since both terms have been divided by the same quantity, the value of the fraction is not changed. (Art. 167, Prin. 3.) And since these terms have no common factor, it follows that $\frac{abd}{3c}$ are the lowest terms required. (Art. 122.)

NOTE.—It will be observed that the factors 5, a, b, and x are prime; therefore, the product $5abx$ is the *g. c. d.* of the numerator and denominator. (Art. 121.) Hence, the

RULE.—*Cancel all the factors common to the numerator and denominator.*

Or, *Divide both terms of the fraction by their greatest common divisor.* (Art. 167.)

2. Reduce $\dfrac{4abcy}{12abc}$ to its lowest terms. *Ans.* $\dfrac{y}{3}$.

3. Reduce $\dfrac{18a^2b}{3ac}$ to its lowest terms. *Ans.* $\dfrac{6ab}{c}$.

---

168. What is reduction of fractions? 169. What are the lowest terms of a fraction? How reduce fractions to the lowest terms?

Reduce the following fractions to the lowest terms:

4. $\frac{3xy}{9x^2y^2}$.

5. $\frac{12a^2bc^3}{4abcd}$.

6. $\frac{17b^2cxy}{51b^2cxy}$.

7. $\frac{84a^2b^2c^3}{108abx^4y^8}$.

8. $\frac{a^2 - b^2}{a^2 + 2ab + b^2}$.

9. $\frac{x^2 - y^2}{x^2 - 2xy + y^2}$.

10. $\frac{3xy^2 - 3x^2y}{2x^2y - 2xyz}$.

11. $\frac{a + bc}{(a + bc) \times x}$.

12. $\frac{x^2 - y^2}{x^4 - y^4}$.

13. $\frac{ax - x^2}{a^2 - x^2}$.

14. $\frac{a - 1}{a^2 - 2a + 1}$.

15. $\frac{x + y}{x^2 + 2xy + y^2}$.

CASE II.

170. To Reduce a Fraction to a *Whole or Mixed* Quantity.

1. Reduce $\frac{2a + 4b + c}{2}$ to a whole or mixed quantity.

OPERATION.

$$\frac{2a + 4b + c}{2} = a + 2b + \frac{c}{2}.$$

ANALYSIS.—Since the value of a fraction is the quotient of the numerator divided by the denominator, it follows that performing the division indicated will give the answer required. Now 2 is contained in $2a$, a times; in $4b$, $2b$ times. Placing the remainder c over the denominator, we have $a + 2b + \frac{c}{2}$, the mixed quantity required. Hence, the

RULE.—*Divide the numerator by the denominator, and placing the remainder over the divisor, annex it to the quotient.*

NOTE.—This rule is based upon the principle that both terms are *divided* by the *same* quantity. (Art. 167, Prin. 3.)

170. How reduce a fraction to a whole or mixed quantity? *Note.* Upon what principle is this rule based?

Reduce the following to whole or mixed quantities:

2. $\frac{ax - x^2}{x}$.

3. $\frac{ab - b^2}{a}$.

4. $\frac{b^2 - c^2}{b + c}$.

5. $\frac{b^3 + c^3}{b - c}$.

6. $\frac{a^2 - 2ab + b^2}{a - b}$.

7. $\frac{b^3 + ab^2}{b^2 - ab}$.

8. $\frac{a^3 + a^2 - ax^2}{a^2 - ax}$.

9. $\frac{12x^2 + 4x - 3y}{4x}$.

CASE III.

171. To Reduce a Mixed Quantity to an *Improper* Fraction.

1. Reduce $a + \frac{b}{3}$ to the form of a fraction.

ANALYSIS.—Since in 1 unit there are three thirds, in a units there must be a times 3 thirds, or $\frac{3a}{3}$; and $\frac{3a}{3} + \frac{b}{3} = \frac{3a+b}{3}$, the fraction required. Hence, the

OPERATION.

$$a + \frac{b}{3} = \frac{3a}{3} + \frac{b}{3}$$
$$\frac{3a}{3} + \frac{b}{3} = \frac{3a + b}{3}$$

RULE.—*Multiply the integer by the denominator; to the product add the numerator, and place the sum over the denominator.*

NOTE.—An integer may be reduced to the form of a fraction by making 1 its denominator. Thus, $a = \frac{a}{1}$.

Reduce the following to improper fractions:

2. $ab - \frac{c}{d}$. *Ans.* $\frac{abd - c}{d}$.

3. $3x + \frac{xy - b}{y}$.

4. $5d + \frac{a - c}{2b}$.

5. $a + b + \frac{2x}{a + b}$.

6. $x - 1 + \frac{1 - x}{1 + x}$.

7. $4a - \frac{a - b}{3c}$.

8. $8x + \frac{3a - x^2}{5x}$.

---

171. How reduce a mixed quantity to an improper fraction? *Note.* An integer?

CASE IV.

172. To Reduce an Integer to a Fraction having *any required Denominator*.

1. Reduce $3a$ to fifths.

ANALYSIS.—Since in $1a$ there are 5 fifths, in $3a$ there must be 3 times 5 fifths, or $\frac{15a}{5}$.

Or, reducing the integer $3a$ to the form of a fraction, it becomes $\frac{3a}{1}$; multiplying both terms by the required denominator, we have $\frac{15a}{5}$. Hence, the

OPERATION.

$$3a = \frac{3a}{1}$$

$$\frac{3a \times 5}{1 \times 5} = \frac{15a}{5}.$$

RULE.—*Multiply the integer by the required denominator, and place the product over it.*

2. Reduce $2x$ to a fraction having $6m$ for its denominator.
3. Reduce $6ax$ to a fraction having $4ab$ for its denominator.
4. Reduce $3a + 4b$ to a fraction having $6c^2$ for its denominator.
5. Reduce $x - y$ to a fraction having $x + y$ for its denominator.
6. Reduce $2x^2y$ to a fraction having $3a^2 - 2b$ for its denominator.

173. To Reduce a Fraction to any Required Denominator.

1. Change $\frac{a}{3}$ to a fraction whose denominator is 12.

ANALYSIS.—Dividing 12, the required denominator, by the given denominator 3, the quotient is 4. Multiplying both terms of the given fraction by the quotient 4, the result, $\frac{4a}{12}$, is the fraction required. Hence, the

OPERATION.

$$12 \div 3 = 4$$

$$Ans.\ \frac{a \times 4}{3 \times 4} = \frac{4a}{12}.$$

RULE.—*Divide the required denominator by the denominator of the given fraction, and multiply both terms by the quotient.*

---

172. How reduce an integer to a fraction having any required denominator?

Reduce the following to the required denominators:

2. Reduce $\frac{3a}{7}$ to thirty-fifths.

SOLUTION. $35 \div 7 = 5$. Now $\frac{3a \times 5}{7 \times 5} = \frac{15a}{35}$. *Ans.*

3. Reduce $\frac{b}{c}$ to the denominator ac.

4. Reduce $\frac{3a}{7}$ to the denominator $49a$.

5. Reduce $\frac{x+y}{x-y}$ to the denominator $x^2 - 2xy + y^2$.

6. Reduce $\frac{4a}{x+y}$ to the denominator $8a^2(x+y)^2$.

COMMON DENOMINATORS.

174. A ***Common Denominator*** is one that belongs equally to *two* or *more* fractions.

PRINCIPLES.

1°. *A common denominator is a multiple of each of the denominators; for every quantity is a divisor of itself and of every multiple of itself.* (Art. 124, Prin. 1.) Hence,

2°. *The least common denominator is the least common multiple of all the denominators.*

CASE V.

175. To Reduce Fractions to Equivalent Fractions having a *Common Denominator*.

1. Reduce $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{x}{y}$, to equivalent fractions having a common denominator.

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173. How reduce a fraction to any required denominator? 174. What is a common denominator? Principle 1? Principle 2?

SOLUTION.—Multiplying the denominators b, d, and y together, we have bdy, which is a common denominator.

$$\begin{aligned} b \times d \times y &= bdy \quad \text{The common denominator.} \\ \left.\begin{aligned} a \times d \times y &= ady \\ c \times b \times y &= bcy \\ x \times b \times d &= bdx \end{aligned}\right\} &\text{The new numerators.} \end{aligned}$$

Ans. $\dfrac{a}{b} = \dfrac{ady}{bdy}, \quad \dfrac{c}{d} = \dfrac{bcy}{bdy}, \quad \dfrac{x}{y} = \dfrac{bdx}{bdy}.$

To reduce the given fractions to this denominator, we multiply each numerator by all the denominators except its own, and place the results over the common denominator. Hence, the

RULE.—*Multiply all the denominators together for a common denominator, and each numerator into all the denominators except its own, for the new numerator.*

NOTES.—1. It is advisable to reduce the fractions to *their lowest* terms, before the rule is applied. (Art. 169.)

2. This rule is based on the principle, that the *value* of a fraction is not changed by multiplying both its terms by the same quantity. (Art. 167, Prin. 3.)

Reduce the following to equivalent fractions having a common denominator:

2. $\dfrac{a}{c}, \dfrac{x}{y}, \dfrac{3}{4}.$ *Ans.* $\dfrac{4ay}{4cy}, \dfrac{4cx}{4cy}, \dfrac{3cy}{4cy}.$

3. $\dfrac{c}{d}, \dfrac{b}{x}, \dfrac{2d}{4}.$

4. $\dfrac{a}{2x}, \dfrac{b}{c^2}, \dfrac{x}{y}.$

5. $\dfrac{2a}{3b}, \dfrac{x}{a+b}.$

6. $\dfrac{x-y}{x+y}, \dfrac{x+y}{x-y}.$

7. $\dfrac{a+b}{3}, \dfrac{5a-1}{a}.$

8. $x, \dfrac{2a}{b}, \dfrac{c+1}{d}.$

9. $\dfrac{2}{3}, \dfrac{a}{b^2}, \dfrac{c+d}{c-d}.$

10. $\dfrac{xy}{z}, \dfrac{1}{2}, \dfrac{2a}{b}.$

11. $\dfrac{2a}{4}, 3, \dfrac{x^2+y^2}{x+y}.$

12. $\dfrac{a-x}{a+x}, \dfrac{a+x}{a-x}.$

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175. How reduce fractions to equivalent fractions having a common denominator? *Note.* Upon what principle is this rule based?

CASE VI.

176. To Reduce Fractions to the *Least* Common Denominator.

1. Reduce $\frac{a}{x}$, $\frac{b}{xy}$, and $\frac{d}{yz}$ to the *l. c. d.*

SOLUTION.—The *l. c. m.* of the denominators is xyz. (Art. 148.)

$$xyz = l.c.d.$$

$$\left.\begin{aligned} xyz \div x &= yz \\ xyz \div xy &= z \\ xyz \div yz &= x \end{aligned}\right\} \text{The multipliers.}$$

$$\left.\begin{aligned} \frac{a \times yz}{x \times yz} &= \frac{ayz}{xyz} \\ \frac{b \times z}{xy \times z} &= \frac{bz}{xyz} \\ \frac{d \times x}{yz \times x} &= \frac{dx}{xyz} \end{aligned}\right\} \text{The fractions required.}$$

To change the given fractions to others whose denominator is xyz, we multiply each numerator by the quotient arising from dividing this multiple by its corresponding denominator. Hence, the

RULE.—I. *Find the least common multiple of all the denominators for the least common denominator.*

II. *Multiply each numerator by the quotient arising from dividing this multiple by its corresponding denominator.*

NOTE.—All the fractions must be reduced to their *lowest* terms before the rule is applied.

Reduce the following fractions to the *l. c. d.*:

2. $\frac{a}{2b}$, $\frac{bc}{x}$, $\frac{y}{4c}$.

3. $\frac{cd}{ab}$, $\frac{2x}{3a}$, $\frac{xy}{ac}$.

4. $\frac{a}{2}$, $\frac{b}{3}$, $\frac{c}{4}$, $\frac{x}{y}$.

5. $\frac{a^2c}{ab}$, $\frac{2cd}{b^2c}$, $\frac{x^2y}{4bc}$.

6. $\frac{2ab}{3ac}$, $\frac{3}{4}$, $\frac{x}{a^2c}$, $\frac{1}{8}$.

7. $\frac{2a}{4b}$, $\frac{cd}{bc}$, $\frac{x^2y}{bcx}$.

8. $\frac{a+b}{a-b}$, $\frac{a-b}{a+b}$, $\frac{a^2+b^2}{a^2-b^2}$.

9. $\frac{2(x+y)}{3(x+y)}$, $\frac{a}{xy}$, $\frac{ab}{6(x+y)}$.

10. $\frac{d}{ab^2}$, $\frac{x}{a^2b}$.

11. $\frac{x}{ac}$, $\frac{m}{b^2c}$, $\frac{y}{c^2d}$.

12. $\frac{x}{y^2}$, $\frac{a+b}{xy}$, $\frac{d}{xz}$.

13. $\frac{m+n}{3a^2}$, $\frac{m-n}{2ax^2}$, $\frac{m^2}{4cx}$.

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176. How reduce fractions to the least common denominator?

ADDITION OF FRACTIONS.

177. When fractions have a *common denominator*, their *numerators* express *like parts* of the *same unit* or *base*, and are *like quantities*. (Art. 43.)

178. To Add Fractions which have a *Common* Denominator.

1. What is the sum of $\frac{3}{8}$, $\frac{7}{8}$, and $\frac{5}{8}$?

SOLUTION. $\frac{3}{8}$ and $\frac{7}{8}$ are $\frac{10}{8}$, and $\frac{5}{8}$ are $\frac{15}{8}$. *Ans.* Hence, the

RULE.—*Add the numerators, and place the sum over the common denominator.*

2. Add $\frac{7b}{m}$, $\frac{4b}{m}$, and $\frac{8b}{m}$. *Ans.* $\frac{19b}{m}$.

3. Add $\frac{3ac}{2xy}$, $\frac{11ac}{2xy}$, $\frac{8ac}{2xy}$, and $\frac{5ac}{2xy}$.

4. Add $\frac{7drz}{5abc}$, $\frac{17dxz}{5abc}$, $\frac{11drz}{5abc}$, and $\frac{4dxz}{5abc}$.

5. Add $\frac{2b+c}{x}$ to $\frac{3b-c}{x}$. 6. Add $\frac{3a+4b}{c}$ to $\frac{4a-2b}{c}$.

179. To Add Fractions which have *Different* Denominators.

7. What is the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{m}{x}$?

ANALYSIS.—Since these fractions have *different* denominators, their numerators cannot be added in their present form. (Art. 66.) We therefore reduce them to a *common denominator*, then add the numerators. (Art. 178.) Hence, the

$$b \times d \times x = bdx, \text{ c. d.}$$

$$\frac{a}{b} = \frac{adx}{bdx} \qquad \frac{c}{d} = \frac{bcx}{bdx}$$

$$\frac{m}{x} = \frac{bdm}{bdx}$$

$$\frac{adx + bcx + bdm}{bdx}, \text{ Ans.}$$

RULE.—*Reduce the fractions to a common denominator, and place the sum of the numerators over it.*

NOTE.—All answers should be reduced to the lowest terms.

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177. When fractions have a common denominator, what is true of the numerators? 178. How add such fractions? 179. How, when they have different denominators?

Find the sum of the following fractions:

8. $\frac{2x}{y}+\frac{2y}{x}+\frac{2xy}{m}$ *Ans.* $\frac{2x^2m+2y^2m+2x^2y^2}{mxy}$.

9. $\frac{3a}{4}+\frac{x}{5}+\frac{y}{3}$.

10. $\frac{2x}{3}+\frac{1}{2a}+\frac{3y}{4}$.

11. $\frac{a}{b+c}+\frac{x}{b-c}$.

12. $\frac{x+y}{2xy}+\frac{x-y}{xy}$.

13. $\frac{2+x}{y}+\frac{3+ax}{ay}$.

14. $\frac{a}{x+y}+\frac{ab}{x-y}$.

15. $\frac{cd}{3x}+\frac{4y}{2d}+\frac{bx}{5}$.

16. $\frac{a}{d}-\frac{2n+d}{3h}$.

17. $\frac{a}{y}+\frac{d}{-m}$.

18. $\frac{-x}{y}+\frac{-h}{m-n}$.

19. $\frac{-4}{2}+\frac{-16}{7-3}$.

20. $\frac{4a}{b}+\frac{6c}{d}-\frac{3m}{3x}$.

180. To Add *Mixed* Quantities.

1. What is the sum of $a+\frac{b}{c}$ and $d-\frac{m}{n}$?

SOLUTION.—Adding the integral and fractional parts separately, the result is $a+d+\frac{bn-cm}{cn}$, the sum required. Hence, the

RULE.—*Add the integral and fractional parts separately, and unite the results.* (Art. 179.)

NOTE.—Mixed quantities may be reduced to improper fractions, and then be added by the rule. (Art. 171.)

2. What is the sum of $a+\frac{b}{2}$ and $c+\frac{d}{x}$?

3. What is the sum of $x+\frac{a}{b}$ and $\frac{-x+d}{m-y}$?

4. What is the sum of $3d-\frac{xy+z}{2}$ and $a+\frac{b-c}{1}$?

5. What is the sum of $5x+\frac{a}{b}$ and $\frac{-y}{2}$?

180. How add mixed quantities? *Note.* How else may they be added?

181. To Incorporate an *Integer* with a Fraction.

6. Incorporate the integer ab with the fraction $\frac{c-d}{3x+y}$.

SOLUTION.—Reducing ab to the denominator of the fraction, we have $ab = \frac{3abx+aby}{3x+y}$; and $\frac{3abx+aby}{3x+y} + \frac{c-d}{3x+y} = \frac{3abx+aby+c-d}{3x+y}$, *Ans.* Hence, the

RULE.—*Reduce the integer to the denominator of the fraction, and place the sum of the numerators over the given denominator.* (Art. 172.)

7. Incorporate the integer $3d$ with the fraction $\frac{2a}{b}$.

8. Incorporate $-4y$ with $\frac{a+b}{c}$.

9. Incorporate $-a$ with $\frac{x+y}{a}$.

10. Incorporate $3x+y$ with $\frac{a-b}{x-y}$.

11. Incorporate $-a+5b$ with $\frac{x-y}{a-b}$.

12. Incorporate $2x+2y$ with $\frac{a+b}{x-1}$.

SUBTRACTION OF FRACTIONS.

182. The *numerators* of fractions which have a *common denominator*, we have seen, are *like quantities.* (Art. 177.) Hence, they may be *subtracted* one from another as integers.

1. Subtract $\frac{5}{8}$ from $\frac{7}{8}$.

SOLUTION. $\frac{7}{8} - \frac{5}{8} = \frac{7-5}{8} = \frac{2}{8}$ or $\frac{1}{4}$. *Ans.* $\frac{1}{4}$.

2. From $\frac{a}{b}$ subtract $\frac{c}{b}$. *Ans.* $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$.

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181. How incorporate an integer with a fraction? 182. What is true of the *numerators* of fractions having a common denominator? How subtract such fractions?

3. From $\frac{13abc}{d}$ subtract $\frac{4abc}{d}$.

4. From $\frac{17xyz}{-a}$ subtract $\frac{5cd}{a}$.

183. To Subtract Fractions which have *Different* Denominators.

5. From $\frac{7a}{3}$ subtract $\frac{3a}{2}$.

ANALYSIS.—Since these fractions have different denominators, they cannot be subtracted one from the other in their present form. We therefore reduce them to a *common denominator*, which is 6, and place the difference of the numerators over it. Hence, the

$$3 \times 2 = 6, \text{ c.d.}$$

$$\frac{7a}{3} = \frac{14a}{6}$$

$$\frac{3a}{2} = \frac{9a}{6}$$

$$\frac{14a}{6} - \frac{9a}{6} = \frac{5a}{6}, \textit{ Ans.}$$

RULE.—*Reduce the fractions to a common denominator, and subtract the numerator of the subtrahend from that of the minuend, placing the difference over the common denominator.*

NOTES.—1. The *integral* and *fractional* parts of mixed quantities should be subtracted separately, and the results be united.

Or, *mixed quantities* may be reduced to improper fractions, and then be *subtracted* by the rule. (Art. 171.)

2. A fraction may be subtracted from an integer, or an integer from a fraction, by reducing the integer to the given denominator, and then applying the rule.

6. From $\frac{5ab}{x}$, take $\frac{3cd}{y}$. *Ans.* $\frac{5aby - 3cdx}{xy}$.

7. From $\frac{a}{m}$, take $\frac{d-b}{y}$.

8. From $\frac{b-d}{m}$, take $-\frac{b}{y}$.

9. From $\frac{a+3d}{4}$, take $\frac{3a-2d}{3}$.

183. How when they have different denominators? *Note* 1. How subtract mixed quantities? *Note* 2. How a fraction from an integer, or an integer from a fraction?

10. From $\frac{h}{m}$, take $-\frac{h+d}{y}$.

11. From $\frac{h}{y}$, take m.

12. From $4a+\frac{b}{c}$, take $3a-\frac{h}{d}$.

13. From $a+\frac{b}{2}$, take $\frac{d-b}{3}$.

14. From $\frac{a}{b-x}$, take $\frac{c}{d+y}$.

15. From $a-\frac{x}{y}$, take $\frac{3d}{2}$.

16. From $\frac{x-y}{10}$, take $\frac{a-b}{x+y}$.

17. From $x-\frac{y-c}{2}$, take $\frac{x-y}{3}-a$.

MULTIPLICATION OF FRACTIONS.

CASE I.

184. To Multiply a Fraction by an *Integer*.

1. Multiply $\frac{a}{b}$ by m.

OPERATION.

$$\frac{a}{b}\times m=\frac{am}{b}.$$

ANALYSIS.—Multiplying the numerator of the fraction by the integer, the product is *am*. (Art. 167, Prin. 1.)

2. What is the product of $\frac{a}{bx}\times x$?

$$\frac{a}{bx}\times x=\frac{a}{bx\div x}$$

$$\frac{a}{bx\div x}=\frac{a}{b}.$$

ANALYSIS.—A fraction is multiplied by dividing its denominator; therefore, if we divide bx by x, the result will be the product required. (Art. 167, Prin. 1.) Hence, the

RULE.—*Multiply the numerator by the integer.*
Or, *Divide the denominator by it.*

NOTES.—1. A fraction is multiplied by a *quantity equal* to its denominator, by *cancelling* the denominator. (Art. 110, Prin. 4.)

184. How multiply a fraction by an integer?

2. A fraction is also multiplied by any *factor* in its *denominator*, by *cancelling that factor.*

Find the product of the following quantities:

3. $\frac{3x}{a-b} \times (a-b)$. *Ans.* $3x$.

4. $\frac{ab}{cd} \times d$. *Ans.* $\frac{ab}{c}$.

5. $\frac{h+3d}{3+m} \times (3+m)$.

6. $\frac{ab}{24} \times 6$.

7. $\frac{2x-3y}{15c+4d} \times (3c+2d)$.

8. $a \times \frac{bc}{3x} \times 6x$.

9. $\frac{a+b}{20x+25xy} \times 5x$.

10. $\frac{a+ab}{bc+c} \times 2ac$.

11. $\frac{2x+3}{5} \times 20x$.

12. $\frac{a-b}{6} \times (12x+18)$.

13. $\frac{abc}{d-x} \times (d-x)$.

14. $\frac{a+b}{20x} \times 4x$.

15. $\frac{2x+y}{40z-10} \times (8z-2)$.

16. $\frac{3c-d}{20} \times 15$.

17. $\frac{3x}{y-1} \times (y^2-1)$.

18. $\frac{m^2}{x^2-z^2} \times (x+z)$.

CASE II.

185. To Multiply a Fraction by a *Fraction*.

1. What is the product of $\frac{a}{c}$ by $\frac{d}{m}$.

ANALYSIS.—Multiplying the numerator of the fraction $\frac{a}{c}$ by d, the numerator of the multiplier, we have $\frac{ad}{c}$. But the multiplier is $\frac{d}{m}$; hence the product $\frac{ad}{c}$ is m times too large.

To correct this, multiply the denominator by m. (Art. 167, Prin. 2.)

OPERATION.

$$\frac{a \times d}{c} = \frac{ad}{c}$$

$$\frac{ad}{c \times m} = \frac{ad}{cm}$$

$$\frac{a}{c} \times \frac{d}{m} = \frac{ad}{cm}$$

Note 1. How is a fraction multiplied by a quantity equal to its denominator?
Note 2. How by any factor in its denominator?

2. Required the product of $\frac{2ab}{6cd}$ multiplied by $\frac{cm}{ax}$.

ANALYSIS.—The factors 2, a, and c are common to each term of the given fractions. Cancelling these common factors, the result is $\frac{bm}{3dx}$, the product required. (Art. 167, Prin. 3.) Hence, the

OPERATION.

$$\frac{2ab}{6cd} \times \frac{cm}{ax} = \frac{2abcm}{6acdx}$$

$$\frac{2abcm}{6acdx} = \frac{bm}{3dx}, \textit{Ans.}$$

RULE.—*Cancel the common factors; then multiply the numerators together for the new numerator, and the denominators for the new denominator.*

NOTES.—1. *Mixed quantities* should be reduced to *improper* fractions, and then be multiplied as above.

Or, the *fractional* and *integral* parts may be multiplied separately, and the results be united.

2. Cancelling the common factors *shortens* the operation, and gives the answer in the *lowest* terms.

3. The word *of* in compound fractions has the force of the sign $\times$. Therefore, reducing compound fractions to simple ones is the same as multiplying the fractions together. Thus, $\frac{2}{5}$ of $\frac{3}{7} = \frac{2}{5} \times \frac{3}{7} = \frac{6}{35}$.

Find the products of the following fractions:

3. $\frac{2x}{y} \times \frac{3xy}{2d} \times \frac{2dy}{x}$.

4. $\frac{bc}{a} \times \frac{x}{by} \times \frac{d}{c}$.

5. $\frac{x-y}{yz} \times \frac{x+y}{y+z}$.

6. $\frac{1}{a+3x} \times \frac{3}{8}$.

7. $\frac{(a+m) \times h}{3x} \times \frac{4y}{(a+m) \times c}$

8. $\frac{a+b}{c^2} \times \frac{cd}{x}$.

9. What is the product of $\frac{3x}{x-y}$ by $\frac{x^2-y^2}{4}$?

SOLUTION.—Factor and cancel. *Ans.* $\frac{3x}{4}(x+y)$.

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185. How multiply a fraction by a fraction? *Note* 1. How mixed quantities? *Note* 2. How shorten the operation? *Note* 3. What is the force of the word *of* in compound fractions?

10. Multiply $\frac{2a}{x-y}$ by $\frac{x^2-y^2}{3d}$. *Ans.* $\frac{2a(x+y)}{3d}$.

11. Multiply $\frac{2x-y}{4x}$ by $\frac{6x-2y}{y^2-2xy}$.

12. Multiply $\frac{4a-2b}{b^2-2ab}$ by $\frac{2a-b}{6a}$.

13. Multiply $a+\frac{a+b}{x}$ by $\frac{ax}{by}$.

14. Multiply $x+\frac{2x^2}{xy}$ by $\frac{x+y}{x^2}$.

15. Multiply $x-\frac{y^2}{x}$ by $\frac{x}{y}+\frac{y}{x}$.

16. Multiply $a+\frac{2a^2}{ab}$ by $\frac{2ab}{a^2}$.

CASE III.

186. To Multiply an Integer by a *Fraction*.

1. Multiply the integer a by $\frac{dx}{cy}$.

ANALYSIS.—Changing the integer to the form of a fraction, we have $\frac{a}{1}$ to be multiplied by $\frac{dx}{cy}$, which equals $\frac{adx}{cy}$. Hence, the

$$a=\frac{a}{1}$$

$$\frac{a}{1}\times\frac{dx}{cy}=\frac{adx}{cy}.$$

RULE.—*Reduce the integer to a fraction; then multiply the numerators together for the new numerator, and the denominators for the new denominator.*

NOTES.—1. Multiplying an *integer* by a *fraction* is the same as finding a *fractional part* of a quantity. Thus, $x\times\frac{3}{4}$ is the same as finding $\frac{3}{4}$ of x, each being equal to $\frac{3x}{4}$. That is,

2. Three times 1 fourth of a quantity is the same as 1 fourth of 3 times that quantity.

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186. How multiply an integer by a fraction? *Note.* To what is this operation similar?

Find the product of the following quantities:

2. $abc \times \frac{dx}{cy}$.

3. $ad \times \frac{b+c}{xy}$.

4. $ax \times \frac{m+n}{4a}$.

5. $(a+h) \times \frac{4c}{d}$.

6. $(3a-y) \times \frac{5x}{y}$.

7. $(x^2+1) \times \frac{5b}{x-1}$.

8. $(1-a^2) \times \frac{7x}{1+a}$.

9. $(x^2-y^2) \times \frac{ac}{3(x+y)}$.

10. $(a^2+ab) \times \frac{3c}{2(a+b)}$.

11. $(x^2+1) \times \frac{2ax}{3(x-1)}$.

12. $2xy(a-b) \times \frac{4x}{a^2-b^2}$.

13. $3a(x-1) \times \frac{2m}{x^2-1}$.

14. $(2ab+b^2) \times \frac{xy}{4a+b}$.

15. $(1-n^2) \times \frac{1}{n+1}$.

187. The principles developed in the preceding cases may be summed up in the following

GENERAL RULE.

Reduce whole and mixed quantities to improper fractions, then cancel the common factors, and place the product of the numerators over the product of the denominators.

1. Multiply $\frac{3c-d}{20}$ by $15x$.

2. Multiply $\frac{3x}{y-1}$ by y^2-1.

3. Multiply $x+\frac{2x^2}{xy}$ by $\frac{x+y}{x^2}$.

4. Multiply $\frac{2ax}{a} \times \frac{3ab}{ac}$ by $\frac{3ac}{2ab}$.

5. Multiply $\frac{3a^2}{10y}$ by $\frac{5y}{9a}$.

6. Multiply $\frac{a^2-b^2}{a} \times \frac{a}{a+b}$ by $\frac{x}{a-b}$.

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187. What is the general rule for multiplying fractions?

7. Multiply $\frac{m^2}{x^2 - z^2}$ by $x + z$.

8. Multiply $\frac{3a^2}{y}$ by $2y^3$.

9. Multiply $\frac{x + y}{x - y}$ by $x^2 - 2xy + y^2$.

10. Multiply $\frac{2x + y}{40z - 10}$ by $8z - 2$.

11. Multiply $x - \frac{y^2}{x}$ by $\frac{x}{y} + \frac{y}{x}$.

12. Multiply $a + \frac{2a^2}{ab}$ by $\frac{2ab}{a^2}$.

13. Multiply $\frac{(c + d)^2}{2a}$ by $\frac{4a^2}{(c + d)}$.

14. Multiply $\frac{2x - y}{4x}$ by $\frac{6x - 2y}{y^2 - 2xy}$.

15. Multiply $b + \frac{2bc}{b - c}$ by $b - \frac{2bc}{b + c}$.

16. Multiply $\frac{2a}{x - y}$ by $\frac{x^2 - y^2}{ax}$.

DIVISION OF FRACTIONS.

CASE I.

188. To Divide a Fraction by an *Integer*.

1. If 3 oranges cost $\frac{9a}{n}$ dollars, what will 1 cost?

OPERATION.

$$\frac{9a}{n} \div 3 = \frac{3a}{n}.$$

ANALYSIS.—One is 1 third of 3; therefore, 1 orange will cost 1 third of $\frac{9a}{n}$ dollars, and $\frac{1}{3}$ of $\frac{9a}{n}$ dollars is $\frac{3a}{n}$ dollars, *Ans.*

2. Divide $\frac{a}{c}$ by m.

OPERATION.

$$\frac{a}{c} \div m = \frac{a}{c \times m} = \frac{a}{cm}.$$

ANALYSIS.—Since we cannot divide the numerator of the fraction by m, we multiply the denominator by it.

The result is $\frac{a}{cm}$. For, in each of the fractions $\frac{a}{c}$ and $\frac{a}{cm}$, the same number of parts is taken; but, since the unit is divided into m times as many parts in the *latter* as in the *former*, it follows that each part in the latter is only $\frac{1}{m}$th of each part in the former. Hence, the

RULE.—*Divide the numerator by the integer.*

Or, *Multiply the denominator by it.*

NOTE.—If the dividend is a *mixed* quantity, it should be reduced to an improper fraction before the rule is applied. (Ex. 3.)

Divide the following quantities:

3. $a + \frac{bc}{x}$ by d. *Ans.* $\frac{ax + bc}{dx}$.

4. $x + \frac{y}{ab}$ by xy. *Ans.* $\frac{abx + y}{abxy}$.

5. $\frac{6x^2y}{n}$ by $3xy$.

6. $\frac{2a^2}{3ac}$ by b.

7. $a + \frac{ab}{c}$ by a.

8. $ax + \frac{axy}{a}$ by x^2.

9. $\frac{a^2 + ax}{2b}$ by $a + x$.

10. $\frac{a^3 - c^3}{b + c}$ by $a - c$.

11. $\frac{x^2 + 2xy + y^2}{a + c}$ by $x + y$.

12. $\frac{x + 2y}{a - b}$ by $a + b$.

CASE II.

189. To Divide a Fraction by a *Fraction*.

This case embraces two classes of examples:

First. Those in which the fractions have a *common* denominator.

Second. Those in which they have *different* denominators.

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188. How divide a fraction by an integer? *Note.* If the dividend is a mixed quantity, how proceed?

1. At $\frac{3a}{m}$ dollars apiece, how many kites can a lad buy for $\frac{12a}{m}$ dollars?

ANALYSIS.—Since these fractions have a *common denominator*, their numerators are *like* quantities, and one may be divided by the other, as integers. (Art. 177.) Now $3a$ is contained in $12a$, 4 times. (Art. 110, Prin. 1.)

OPERATION.

$$\frac{12a}{m} \div \frac{3a}{m} = 4$$

Ans. 4 kites.

2. How many times is $\frac{5ab}{x}$ contained in $\frac{15ab}{x}$?

3. What is the quotient of $\frac{35bc}{2ad}$ divided by $\frac{7bc}{2ad}$.

4. It is required to divide $\frac{a}{x}$ by $\frac{c}{y}$.

ANALYSIS.—Since these fractions have *different* denominators, their numerators are *unlike* quantities; consequently, one cannot be divided by the other in this form. (Art. 114, *note*.) We therefore reduce them to a *common denominator*; then dividing one numerator by the other, the result is the quotient.

FIRST OPERATION.

$$\frac{a}{x} = \frac{ay}{xy}, \quad \frac{c}{y} = \frac{cx}{xy},$$

$$\frac{ay}{xy} \div \frac{cx}{xy} = \frac{ay}{cx}.$$

SECOND OPERATION.

$$\frac{a}{x} \div \frac{c}{y} = \frac{a}{x} \times \frac{y}{c} = \frac{ay}{cx}, \quad \textit{Ans.}$$

Or, more briefly, if we *invert* the divisor, and multiply the dividend by it, we have the *same combinations* and the *same result* as before. (Art. 185.) Hence, the

RULE.—*Multiply the dividend by the divisor inverted.*

Or, *Reduce the fractions to a common denominator, and divide the numerator of the dividend by that of the divisor.*

NOTES.—1. A fraction is *inverted*, when its terms are made to change places. Thus, $\frac{a}{b}$ inverted, becomes $\frac{b}{a}$.

2. The object of inverting the divisor is convenience in multiplying.

3. After the divisor is inverted, the *common factors* should be *cancelled* before the multiplication is performed.

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189. How divide a fraction by a fraction when they have a common denominator? When the denominators are different, how?

Divide the following fractions:

5. $\frac{ab}{cd}$ by $\frac{x}{ay}$.

6. $\frac{3abx}{6aby}$ by $\frac{2y}{ax}$.

7. $\frac{3x^2}{4}$ by $\frac{xy}{2}$.

8. $\frac{xy}{x-1}$ by $\frac{xy}{2}$.

9. $\frac{a-1}{x}$ by $\frac{2}{ax}$.

10. $\frac{5x^2y^2}{10ab}$ by $\frac{15xy}{20abx}$.

11. $\frac{12(x+y)}{ab}$ by $\frac{4(x+y)}{2ab}$.

12. $\frac{3a^2b^2}{a+b}$ by $\frac{6ab}{a}$.

13. $\frac{x+y}{2b^2c^2}$ by $\frac{axy}{6bc}$.

14. $\frac{x-a}{4cd}$ by $\frac{3bc}{2d}$.

15. $\frac{2xy}{x+y}$ by $\frac{3ax}{4y}$.

16. $\frac{2x^2y^2}{a+b}$ by $\frac{3xyz}{2y}$.

17. $\frac{x^4}{ax^2}$ by $\frac{ax}{bx}$.

18. $\frac{36ad}{5b}$ by $\frac{18ab}{10by}$.

19. $\frac{5b}{36ad}$ by $\frac{10by}{18ab}$.

20. $\frac{10by}{18ab}$ by $\frac{5b}{36ad}$.

CASE III.

190. To Divide an Integer by a *Fraction*.

1. Divide the integer $7dc$ by $\frac{3ad}{b}$.

OPERATION.

$$7dc \div \frac{3ad}{b} =$$

$$\frac{7dc}{1} \times \frac{b}{3ad} = \frac{7bc}{3a}, \text{ Ans.}$$

ANALYSIS.—Having reduced the integer to a fraction, and inverted the divisor, we *cancel* the *common* factor d, and proceed as in the last case. Hence, the

RULE.—*Reduce the integer to a fraction, and multiply it by the divisor inverted.*

Divide the following quantities:

2. $aby \div \frac{cx}{dm}$.

3. $ax \div \frac{4a}{m+n}$.

4. $(a+x) \div \frac{5c}{x}$.

5. $(5x-y) \div \frac{y}{5x}$.

6. $(x^2+1) \div \frac{x+1}{5a}$.

7. $(1-a^2) \div \frac{1+a}{3x}$.

190. How divide an integer by a fraction?

191. ***Complex Fractions*** are reduced to *simple* ones, by performing the *division indicated.*

8. Reduce $\dfrac{\frac{a}{b}}{\frac{3}{4}}$ to a simple fraction.

ANALYSIS.—The given fraction is equivalent to $\frac{a}{b} \div \frac{3}{4}$. Performing the division indicated, we have $\frac{4a}{3b}$, the simple fraction required.

$\frac{a}{b}$, the dividend.

$\frac{3}{4}$, the divisor.

$\frac{a}{b} \times \frac{4}{3} = \frac{4a}{3b}$, *Ans.*

Reduce the following fractions to simple ones:

9. $\dfrac{\frac{a^2}{b}}{cd}$. *Ans.* $\frac{a^2}{bcd}$.

10. $\dfrac{\frac{a+1}{a-1}}{\frac{a-1}{a+1}}$.

12. $\dfrac{x^2-y^2}{\frac{a-b}{x+y}}$.

11. $\dfrac{\frac{a-b}{x+y}}{\frac{x-y}{a+b}}$.

13. $\dfrac{\frac{x+y}{a-b}}{\frac{a+b}{x-y}}$.

192. The various principles developed in the preceding cases may be summed up in the following

GENERAL RULE.

Reduce integers and mixed quantities to improper fractions, and complex fractions to simple ones: then multiply the dividend by the divisor inverted.

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191. How reduce complex fractions? 192. What is the general rule for dividing fractions?

1. Divide $\frac{15abc}{4xyz}$ by $3bc$.

2. Divide $\frac{35bcd}{27y}$ by $9xy$.

3. Divide $\frac{16xy}{13a}$ by $\frac{2cd}{39a}$.

4. Divide $\frac{42ab}{x^2 - y^2}$ by $\frac{14b}{x - y}$.

5. Divide $\frac{23xyz}{a^2 + 2ab + b^2}$ by $\frac{17y^2}{a + b}$.

6. Divide $\frac{x^2 - 2xy + y^2}{3bc}$ by $\frac{x - y}{24b^2}$.

7. Divide $\frac{a^4 - m^4}{a^2 - 2am + m^2}$ by $\frac{a^2 + am}{a - m}$.

SOLUTION.—Factoring and cancelling, we have,

$\frac{a^4-m^4}{a^2-2am+m^2} = \frac{(a^2+m^2)(a+m)(a-m)}{(a-m)(a-m)}$; $\frac{a^2+am}{a-m} = \frac{a(a+m)}{a-m}$. Then,

$\frac{(a^2+m^2)(a+m)(a-m)}{(a-m)(a-m)} \times \frac{a-m}{a(a+m)} = \frac{a^2+m^2}{a}$, or $a + \frac{m^2}{a}$, *Ans.*

8. Divide $\frac{3a}{a^2 - x^2}$ by $\frac{3}{a - x}$.

9. Divide $\frac{4c^2 - 8c}{x + y}$ by $\frac{c^2 - 4}{x + y}$.

10. Divide $\frac{d^2 - dx}{ac + ax}$ by $\frac{3(c - x)}{4(d + x)}$.

11. Divide $\frac{2c^2}{a^3 + c^3}$ by $\frac{c}{a + c}$.

12. Divide $\frac{4(a^2 - x^2)}{x}$ by $\frac{3(a + x)}{a - x}$.

13. Divide $\frac{a - b}{a^2 + 2ab + b^2}$ by $\frac{a^2 - b^2}{a + b}$.

14. Divide $\frac{x}{x^2 - 1}$ by $\frac{x + 1}{x - 1}$.

15. Divide $\frac{bc^2 + bcd}{x + ax}$ by $\frac{b(c + d)}{1 + a}$.

CHAPTER IX.

SIMPLE EQUATIONS.

193. An ***Equation*** is an expression of equality between two quantities. (Art. 27.)

194. Every equation consists of *two parts,* called the *first* and *second members.*

195. The ***First Member*** is the part on the *left* of the sign (=).

The ***Second Member*** is the part on the *right* of the sign (=).

196. Equations are divided into *degrees,* according to the exponent of the unknown quantity; as the first, second, third, fourth, etc.

Equations are also divided into *Simple, Quadratic, Cubic,* etc.

197. A ***Simple Equation*** is one which contains only the *first* power of the unknown quantity, and is of the *first* degree; as, $ax = d$.

198. A ***Quadratic Equation*** is one in which the *highest* power of the unknown quantity is a *square,* and is of the *second* degree; as, $ax^2 + cx = d$.

199. A ***Cubic Equation*** is one in which the *highest* power of the unknown quantity is a *cube,* and is of the *third* degree; as, $ax^3 + bx^2 - cx = d$.

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193. What is an equation? 194. How many parts? 195. Which is the first member? The second? 196. How are equations divided? What other divisions? 197. What is a simple equation? 198. A quadratic? 199. Cubic?

200. An ***Identical Equation*** is one in which both members have the *same form*, or may be reduced to the same form.

Thus, $ab-c = ab-c$, and $8x-3x = 5x$, are identical.

NOTE.—Such an equation is often called an *identity*.

201. The ***Transformation*** of an equation is changing its *form* without destroying the *equality* of its members.

NOTE.—The members of an equation will retain their *equality*, so long as they are *equally increased* or *diminished*. (Ax. 2–5.)

TRANSPOSITION.

202. ***Transposition of Terms*** is changing them from one side of an equation to the other without destroying the equality of its members.

203. ***Unknown Quantities*** may be combined with *known* quantities by addition, subtraction, multiplication, or division.

NOTE.—The *object* of transposition is to obtain an equation in which the *terms* containing the *unknown* quantity shall stand on one side, and the *known* terms on the other.

204. To *Transpose* a Term from one Member of an Equation to the other.

1. Given $x + b = a$, to find the value of x.

SOLUTION.—By the problem, $x+b = a$

Adding $-b$ to each side (Ax. 2), $x+b-b = a-b$

Cancelling $(+b-b)$ (Ax. 7), $\therefore\ x = a-b$

This result is the same as *changing the sign* of b from + to − in the first equation, and transposing it to the other side.

2. Given $x - d = c$, to find the value of x.

SOLUTION.—By the problem, $x-d = c$

Adding $+d$ to each side (Ax. 2), $x-d+d = c+d$

Cancelling $(-d+d)$, (Ax. 7.) $\therefore\ x = c+d$

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200. Identical? 201. What is the transformation of an equation? *Note*. Equality. 202. What is transposition of terms? 203. How combine unknown quantities? *Note*. Object of transposition?

This result is also the same as *changing the sign* of d from $-$ to $+$, and transposing it to the other side. Hence, the

RULE.—*Transpose the term from one member of the equation to the other, and change its sign.*

NOTE.—In the *first* of the preceding examples, the unknown quantity is combined with one that is known by *addition;* in the *second*, with one by *subtraction*.

3. Given $b - c + x = a - d$, to find x.
4. Given $x + ab - c = a + b$, to find x.

205. The ***Signs*** of all the terms of an equation may be changed *without destroying* the *equality*. For, all the terms on each side may be transposed to the other, by changing their signs.

206. If *all the terms* on one side are transposed to the other, *each member* will be equal to o.

Thus, if $x+c=d$, it follows that $x+c-d=0$.

REDUCTION OF EQUATIONS.

207. The ***Reduction*** of an equation consists in finding the *value* of the unknown quantity which it contains.

208. The ***Value*** of an unknown quantity is the *number* which, substituted for it, will satisfy the equation. Hence, it is sometimes called the *root* of the equation.

209. The reduction of equations depends on the following

PRINCIPLE.

Both members of an equation may be increased or diminished by the same quantity without destroying the equality.

204. How transpose a term from one member of an equation to the other? 205. What is the effect of changing all the signs? 206. Of transposing all the terms? 207. In what does the reduction of an equation consist? 208. What is the value of an unknown quantity? What sometimes called? 209. Upon what principle does the reduction of equations depend?

210. This principle may be illustrated by a pair of scales. If 4 balls, each weighing 1 lb., are placed in each scale, they balance each other.

Adding 2 lbs. to each scale,

$$4+2=4+2$$

Subtracting 2 lbs. from each,

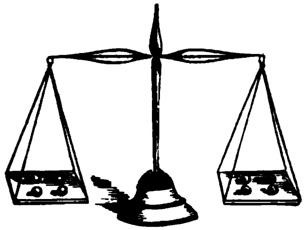
$$4-2=4-2$$

Multiplying each by 2,

$$4\times 2=4\times 2$$

Dividing each by 2,

$$4\div 2=4\div 2$$



211. To Reduce an Equation containing One Unknown Quantity by *Transposition*.

5. Given $2x-3a+7=x+35$, to find x.

SOLUTION.—By the problem, $2x-3a+7=x+35$

Transposing the terms (Art. 204), $2x-x=35-7+3a$

Uniting the terms, *Ans.* $x=28+3a$

Therefore, $28+3a$ is the value of x required. Hence, the

RULE.—*Transpose the unknown quantities to one side, and the known quantities to the other, and unite the terms.*

NOTES.—1. Transposing the terms is the same, in effect, as *adding* equal quantities to, or *subtracting* them from each member; hence, it is often called reduction of equations by *addition* or *subtraction*. (Arts. 72, 75, Prin. 4.)

2. Uniting the terms depends upon Axiom 9.

212. When the *same term*, having the *same sign*, is on *opposite* sides of the equation, it may be *cancelled.*

6. Reduce $3x+a-6=b-4+2x$.
7. Reduce $x-3+c=2x+a-b$.
8. Reduce $2y+bc-ad=y+2m-8$.
9. Reduce $3ab-y+d=-2y+17$.
10. Reduce $4cd+27-4x+d=28-3x+3bh$.
11. Reduce $b+c-4x=32+b-5x+d$.
12. Reduce $x+4-2x-3=3x+4+8-5x$.

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210. Illustrate this principle? 211. What is the rule for reducing equations? *Note.* To what is transposition equivalent? 212. When the *same term*, having the *same sign*, is on opposite sides, what may be done?

CLEARING OF FRACTIONS.

213. To Reduce an Equation containing Fractions.

1. Given $\frac{x}{2} + \frac{2x}{6} = 27$, to find the value of x.

SOLUTION.—By the problem, $\frac{x}{2} + \frac{2x}{6} = 27$

Multiplying each term by 6, the *l. c. m.* of the denominators (Art. 148), $3x + 2x = 162$

Uniting the terms, $5x = 162$

Dividing each side by the coefficient, *Ans.* $x = 32\frac{2}{5}$

2. Given $\frac{x}{2} + \frac{x}{3} = \frac{30}{4}$, to find the value of x.

SOLUTION.—By the problem, $\frac{x}{2} + \frac{x}{3} = \frac{30}{4}$

Mult. by 12, the *l. c. m.* of denominators, $6x + 4x = 90$

Uniting terms, and dividing (Art. 211), *Ans.* $x = 9$

Therefore, the value of x is 9. Hence, the

RULE.—*Multiply each term of the equation by the least common multiple of the denominators; then, transposing and uniting the terms, divide each member by the coefficient of the unknown quantity.*

NOTES.—1. An equation may also be *cleared of fractions*, by multiplying both sides by *each denominator* separately.

2. The *reason* that clearing an equation of fractions does not destroy the equation, is because both members are *multiplied* by the same quantity. (Ax. 4.)

3. A fraction is *multiplied* by its *denominator* by *cancelling* the denominator. (Art. 184, *Note* 1.)

4. Removing the *coefficient* of a quantity *divides* the quantity by it.

5. If any given numerator is a *multiple* of its denominator, divide the *former* by the *latter* before applying the rule.

6. The unknown quantity in the last two problems is combined with those that are known by *multiplication* and *division*. Hence, the operation is often called, reduction of equations by *multiplication* and *division*.

---

213. Rule for clearing an equation of fractions? *Notes.* 1. In what other way may fractions be removed? 2. Why does not this process destroy the equation? 3. What is the effect of cancelling a denominator? 4. Effect of removing a coefficient? 5. If a numerator is a multiple of its denominator, how proceed?

3. Reduce $\frac{3x}{5} + 12 = \frac{4x}{3} + 1$.

4. Reduce $\frac{2x}{3} - \frac{x}{6} = 6x - 66$.

5. Reduce $\frac{4x}{10} + \frac{3}{5} = 35 - x$.

214. When the sign $-$ is prefixed to a fraction and the denominator is removed, the *signs of all the terms* in the numerator must be changed from $+$ to $-$, or $-$ to $+$.

6. Reduce $3x - \frac{x-2}{5} = 20$.

Solution.—By the problem, $3x - \frac{x-2}{5} = 20$

Removing the denominator 5, $15x - x + 2 = 100$

Uniting and transposing the terms, $14x = 98$

Dividing by the coefficient, *Ans.* $x = 7$

7. Given $\frac{4a - 5b}{x} = -\frac{3b}{d}$, to find x.

8. Given $3x - \frac{7x}{5} = a - \frac{2b + c}{10}$, to find x.

9. Given $-x + \frac{x}{3} + \frac{3x}{4} = \frac{15}{24}$, to find x.

10. Given $a + b + c = \frac{x}{2} + \frac{x}{4} + a + b + c \div 5$, to find x.

215. The principles developed by the preceding illustrations may be summed up in the following

GENERAL RULE.

I. *Clear the equation of fractions.* (Art. 213.)

II. *Transpose and unite the terms.* (Art. 204.)

III. *Divide both sides by the coefficient of the unknown quantity.* (Art. 213, *Note* 4.)

Proof.—*For the unknown quantity substitute its value, and if it satisfies the equation, the work is right.*

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214. If sign $-$ is prefixed to a fraction? 215. What is the general rule for simple equations? How proved?

EXAMPLES.

1. Given $x + \frac{x}{2} + \frac{x}{4} = 14$, to find x.

2. Given $\frac{x}{2} + x = \frac{7x}{10} + 40$, to find x.

3. Given $\frac{4x}{5} + 10 = \frac{7x}{10} + 13$, to find x.

4. Given $\frac{14}{x-2} + 6 = 8$, to find x.

5. Given $x + \frac{2x+1}{5} = 10$, to find x.

6. Given $2x + \frac{6x+11}{5} = 18 + \frac{9x-29}{4}$, to find x.

7. Given $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 78$, to find x.

8. Given $\frac{4x-6}{6} - 8 = \frac{x-6}{4} + 5$, to find x.

9. Given $x + \frac{3x-5}{2} + \frac{4x-8}{6} = 12$, to find x.

10. Given $2x - 16 = \frac{4x+8}{3}$, to find x.

11. Given $\frac{2x-8}{4} + \frac{x+32}{2} + \frac{x}{3} = 30$, to find x.

12. Given $\frac{x}{2} + \frac{2x}{6} = 16 + \frac{x}{6}$, to find x.

13. Given $\frac{3x+1}{2} - 10 = \frac{2x}{3} + \frac{x-1}{6}$, to find x.

14. Given $\frac{6x}{10} + \frac{5x}{6} = \frac{4x}{15} - 1\frac{4}{15}$, to find x.

15. Given $8x + 6\frac{1}{4} - \frac{x}{2} = 8\frac{1}{4} - \frac{2x}{7} + \frac{15x}{2}$, to find x.

Note.—Sometimes there is an advantage in uniting similar terms, before clearing of fractions. Thus, uniting $6\frac{1}{4}$ with $8\frac{1}{4}$; also $-\frac{x}{2}$ with $\frac{15x}{2}$, we have, $8x = 2 - \frac{2x}{7} + 8x$; $\therefore\ x = 7$, *Ans.*

16. Given $\frac{3x}{6} - 6 + \frac{2x}{6} = \frac{x}{6} + 2$, to find x.

17. Given $\frac{4x}{5} = \frac{3x}{4} + 15 - 12$, to find x.

18. Given $2x - 4 = \frac{x}{2} + 2$, to find x.

19. Given $\frac{3x}{4} + \frac{4x}{5} - 1\frac{5}{6} = \frac{3x}{5} + 17\frac{1}{6}$, to find x.

20. Given $\frac{x}{a} = b + c$, to find x.

21. Given $\frac{ax}{n} = d$, to find x.

22. Given $\frac{ax}{2} + \frac{bx}{3} = c$, to find x.

23. Given $\frac{2ax + b}{a} = \frac{cx + d}{c}$, to find x.

24. Given $\frac{2c}{a} + \frac{b}{x} = \frac{c}{2} + \frac{c}{a}$, to find x.

25. Given $\frac{2a}{3} + \frac{4x}{5} + \frac{3b}{2} = \frac{5a}{6} + \frac{12b}{2}$, to find x.

26. Given $\frac{3x}{2} + \frac{2a}{3} = \frac{4b}{5} + \frac{15c}{3}$, to find x.

27. Given $\frac{3a + x}{x} - 5 = \frac{6}{x}$, to find x.

28. Given $\frac{x - 1}{x + 1} + 1 = \frac{1}{a}$, to find x.

29. Given $\frac{x}{a} + \frac{x}{c - a} = \frac{a}{c} + a$, to find x.

30. Given $x + b = \frac{x^2}{x + b}$, to find x.

31. Given $x - a = \frac{x^2 + a}{x - a}$, to find x.

32. Given $3\left(\frac{x + b}{4}\right) + \left(\frac{x - b}{3}\right) = 4\left(\frac{x - b}{3}\right)$, to find x.

33. Given $\frac{5x}{9} = x - \frac{2x - 56}{3}$, to find x.

34. Given $\frac{3x}{4} + x - \frac{x}{2} = 25$, to find x.

35. Given $80 = 4x - \frac{x}{2} - \frac{x}{6}$, to find x.

36. Given $\frac{2x+1}{3} = 2x - \frac{x+3}{4}$, to find x.

37. Given $10 - 2x = \frac{3x+4}{3} - \frac{24-36}{3}$, to find x.

38. Given $x - 3 = 15 - \frac{x+4}{11}$, to find x.

39. Given $x + 2 = 3x + \frac{x+8}{4} - \frac{x+6}{3}$, to find x.

40. Given $\frac{3x}{4} + \frac{x-4}{2} - \frac{x-10}{2} = x - 6$, to find x.

41. Given $\frac{11x-1}{12} = \frac{5x-11}{4} - \frac{x-1}{10}$, to find x.

42. Given $\frac{4x}{5} - \frac{7x}{10} = 120$, to find x.

43. Given $x - 20 = -\frac{2x+1}{5}$, to find x.

44. Given $\frac{3x-5}{2} = \frac{4-2x}{3} + 12 - x$, to find x.

45. Given $\frac{1-x}{6} + 10 = \frac{2x}{3} - \frac{1-3x}{2}$, to find x.

46. Given $\frac{x}{a+1} - \frac{x}{a-1} = b$, to find x.

47. Given $\frac{x}{a-b} - \frac{2+x}{a+b} = \frac{c}{a^2-b^2}$, to find x.

48. Given $\frac{3a+x}{x} = 5 + \frac{6}{x}$, to find x.

49. Given $\frac{bx}{2} = d - \frac{bx}{3}$, to find x.

50. Given $8a = \frac{1-x}{1+x}$, to find x.

51. Given $\frac{x^2+4x+4}{x+2} = \frac{4ab}{16b}$, to find x.

PROBLEMS.

216. The ***Solution*** of a problem is finding a *quantity* which will satisfy its *conditions.* It consists of *two parts:*

First.—The ***Formation*** of an equation which will express the conditions of the problem in algebraic language.

Second.—The ***Reduction*** of this equation.

217. To Solve *Problems* in Simple Equations containing one unknown Quantity.

1. A farmer divided 52 apples among 3 boys in such a manner that B had 1 half as many as A, and C 3 fourths as many as A minus 2. How many had each?

1. FORMATION—Let $x =$ A's number.

By the conditions, $\frac{x}{2} =$ B's "

" " $\frac{3x}{4} - 2 =$ C's "

Therefore, by Ax. 9, $x + \frac{x}{2} + \frac{3x}{4} - 2 = 52$, the whole.

2. REDUCTION— $4x + 2x + 3x - 8 = 208$

Transposing, etc., $9x = 216$

Removing the coefficient, $x = 24$, A's number.

Ans. A had 24, B had 12, and C had $18 - 2 = 16$.

From this illustration we derive the following

GENERAL RULE.

I. *Represent the unknown quantity by a letter, then state in algebraic language the operations necessary to satisfy the conditions of the problem.*

II. *Clear the equation of fractions; then, transposing and uniting the terms, divide each member by the coefficient of the unknown quantity.* (Art. 213.)

NOTE.—A *careful study of the conditions* of the problem will soon enable the learner to *discover* the *quantity* to be represented by the letter, and the *method* of forming the equation.

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216. What is the solution of a problem? Of what does it consist? 217. What is the general rule?

2. The bill for a coat and vest is \$40; the value of the coat is 4 times that of the vest. What is the value of each?

3. A bankrupt had \$9000 to pay A, B, and C; he paid B twice as much as A, and C as much as A and B. What did each receive?

4. The whole number of hands employed in a factory was 1000; there were twice as many boys as men, and 11 times as many women as boys. How many of each?

5. Two trains start at the same time, at opposite ends of a railroad 120 miles long, one running twice as fast as the other. How far will each have run at the time of meeting?

6. A man bought equal quantities of two kinds of flour, at \$10 and \$8 a barrel. How many barrels did he buy, the whole cost being \$1200?

7. If 96 pears are divided among 3 boys, so that the second shall have 2, and the third 5, as often as the first has 1, how many will each receive?

8. A post is one-fourth of its length in the mud, one-third in the water, and 12 feet above water; what is its whole length?

9. After paying away $\frac{1}{4}$ of my money, and then $\frac{1}{5}$ of the remainder, I have \$72. What sum had I at first?

10. Divide \$300 between A, B, and C, so that A may have twice as much as B, and C as much as both the others.

11. At the time of marriage, a man was twice as old as his wife; but after they had lived together 18 years, his age was to hers as 3 to 2. Required their ages on the wedding day.

12. A and B invest equal amounts in trade. A gains \$1260 and B loses \$870; A's money is now double B's. What sum did each invest?

13. Required two numbers whose difference is 25, and twice their sum is 114.

14. A merchant buying goods in New York, spends the first day $\frac{1}{3}$ of his money; the second day, $\frac{1}{4}$; the third day, $\frac{1}{5}$; the fourth day, $\frac{1}{6}$; and he then has \$300 left. How much had he at first?

15. What number is that, from the triple of which if 17 be subtracted the remainder is 22?

16. In fencing the side of a field whose length was 450 rods, two workmen were employed, one of whom built 9 rods and the other 6 rods per day. How many days did they work?

17. Two persons, 420 miles apart, take the cars at the same time to meet each other; one travels at the rate of 40 miles an hour, and the other at the rate of 30 miles. What distance does each go?

18. Divide a line of 28 inches in length into two such parts that one may be $\frac{3}{4}$ of the other.

19. Charles and Henry have $200, and Charles has seven times as much money as Henry. How much has each?

20. What is the time of day, provided $\frac{3}{7}$ of the time past midnight equals the time to noon?

21. A can plow a field in 20 days, B in 30 days, and C in 40 days. In what time can they together plow it?

22. A man sold the same number of horses, cows, and sheep; the horses at $100, the cows at $45, and the sheep at $5, receiving $4800. How many of each did he sell?

23. Divide 150 oranges among 3 boys, so that as often as the first has 2, the second shall have 5, and the third 3. How many should each receive?

24. Four geese, three turkeys, and ten chickens cost $10; a turkey cost twice as much as a goose, and a goose 3 times as much as a chicken. What was the price of each?

25. The head of a fish is 4 inches long; its tail is 12 times as long as its head, and the body is one-half the whole length. How long is the fish?

26. Divide 100 into two parts, such that one shall be 20 more than the other.

27. Divide a into two such parts, that the greater divided by c shall be equal to the less divided by d.

28. How much money has A, if $\frac{1}{4}$, $\frac{2}{3}$, and $\frac{1}{6}$ of it amount to $1222?

29. What number is that, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ of which are equal to 60?*

30. A man bought beef at 25 cents a pound, and twice as much mutton at 20 cents, amounting to \$39. How many pounds of each?

31. A says to B, "I am twice as old as you, and if I were 15 years older, I should be 3 times as old as you." What were their ages?

32. The sum of the ages of A, B, and C is 110 years; B is 3 years younger than A, and 5 years older than C. What are their ages?

33. At an election, the successful candidate had a majority of 150 votes out of 2500. What was his number of votes?

34. In a regiment containing 1200 men, there were 3 times as many cavalry as artillery less 20, and 92 more infantry than cavalry. How many of each?

35. Divide \$2000 among A, B, and C, giving A \$100 more than B, and \$200 less than C. What is the share of each?

36. A prize of \$150 is to be divided between two pupils, and one is to have $\frac{2}{3}$ as much as the other. What are the shares?

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* When the conditions of the problem contain fractional expressions, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., we can avoid these fractions, and greatly abridge the operation, by representing the quantity sought by such a number of x's as can be divided by each of the denominators without a remainder. This number is easily found by taking the least common multiple of all the denominators. Thus, in problem 29,

Let	$12x =$ the number.
Then will	$6x = 1$ half,
" "	$4x = 1$ third,
" "	$3x = 1$ fourth,
" "	$2x = 1$ sixth,
Hence,	$6x + 4x + 3x + 2x = 60$
	$\therefore \ x = 4$
Finally,	$x \times 12$ or $12x = 48$, the number required.

37. Two horses cost \$616, and 5 times the cost of one was 6 times the cost of the other. What was the price of each?

38. What were the ages of three brothers, whose united ages were 48 years, and their birthdays 2 years apart?

39. A messenger travelling 50 miles a day had been gone 5 days, when another was sent to overtake him, travelling 65 miles a day. How many days were required?

40. What number is that to which if 75 be added, $\frac{2}{5}$ of the sum will be 250?

41. It is required to divide 48 into two parts, which shall be to each other as 5 to 3.*

42. What quantity is that, the half, third, and fourth of which is equal to a?

43. A and B together bought 540 acres of land, and divided it so that A's share was to B's as 5 to 7. How many acres had each?

44. A cistern has 3 faucets; the first will empty it in 6 hours, the second in 10, and the third in 12 hours. How long will it take to empty it, if all run together?

45. Divide the number 39 into 4 parts, such that if the first be increased by 1, the second diminished by 2, the third multiplied by 3, and the fourth divided by 4, the results will be equal to each other.

46. Find a number which, if multiplied by 6, and 12 be added to the product, the sum will be 66.

47. A man bought sheep for \$94; having lost 7 of them, he sold $\frac{1}{4}$ of the remainder at cost, receiving \$20. How many did he buy?

48. A and B have the same income; A saves $\frac{1}{4}$ of his, but B spending \$50 a year more than A, at the end of 5 years is \$100 in debt. What is their income?

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* When the quantities sought have a given ratio to each other, the solution may be abridged by taking such a number of x's for the unknown quantity, as will express the ratio of the quantities to each other without fractions. Thus, taking $5x$ for the first part, $3x$ will represent the second part; then $5x + 3x = 48$, etc.

49. A cistern is supplied with water by one pipe and emptied by another; the former fills it in 20 minutes, the latter empties it in 15 minutes. When full, and both pipes run at the same time, how long will it take to empty it?

50. What number is that, if multiplied by m and n separately, the difference of their products shall be d?

51. A hare is 50 leaps before a greyhound, and takes 4 leaps to the hound's 3 leaps; but 2 of the greyhound's equal 3 of the hare's leaps. How many leaps must the hound take to catch the hare?

52. What two numbers, whose difference is b, are to each other as a to c?

53. A fish was caught whose tail weighed 9 lbs.; his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail together. What was the weight of the fish?

54. An express messenger travels at the rate of 13 miles in 2 hours; 12 hours later, another starts to overtake him, travelling at the rate of 26 miles in 3 hours. How long and how far must the second travel before he overtakes the first?

55. A father's age is twice that of his son; but 10 years ago it was 3 times as great. What is the age of each?

56. What number is that of which the fourth exceeds the seventh part by 30?

57. Divide $576 among 3 persons, so that the first may have three times as much as the second, and the third one-third as much as the first and second together.

58. In the composition of a quantity of gunpowder, the nitre was 10 lbs. more than $\frac{2}{3}$ of the whole, the sulphur $4\frac{1}{2}$ lbs. less than $\frac{1}{6}$ of the whole, the charcoal 2 lbs. less than $\frac{1}{7}$ of the nitre. What was the amount of gunpowder? *

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* The operation will be shortened by the following artifice:

Let $42x+48$ = the number of pounds of powder.

Then $28x+42$ = nitre; $7x+3\frac{1}{2}$ = sulphur; $4x+4$ = charcoal.

Hence, $39x+49\frac{1}{2} = 42x + 48$.

$\therefore\ x = \frac{1}{2}$, and $42x+48 = 69$ pounds, *Ans.*

59. Divide 36 into 3 parts, such that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third are all equal to each other.

60. Divide a line 21 inches long into two parts, such that one may be $\frac{3}{4}$ of the other.

61. A milliner paid \$5 a month for rent, and at the end of each month added to that part of her money which was not thus spent a sum equal to 1 half of this part; at the end of the second month her original money was doubled. How much had she at first?

62. A man was hired for 60 days, on condition that for every day he worked he should receive 75 cents, and for every day he was absent he should forfeit 25 cents; at the end of the time he received \$12. How many days did he work?

63. Divide \$4200 between two persons, so that for every \$3 one received, the other shall receive \$5.

64. A father told his son that for every day he was perfect in school he would give him 15 cents; but for every day he failed he should charge him 10 cents. At the end of the term of 12 weeks, 60 school days, the boy received \$6. How many days did he fail?

65. A young man spends $\frac{1}{3}$ of his annual income for board, $\frac{1}{8}$ for clothing, $\frac{1}{10}$ in charity, and saves \$318. What is his income?

66. A certain sum is divided so that A has \$30 less than $\frac{1}{2}$, B \$10 less than $\frac{1}{3}$, and C \$8 more than $\frac{1}{4}$ of it. What does each receive, and what is the sum divided?

67. The ages of two brothers are as 2 to 3; four years hence they will be as 5 to 7. What are their ages?*

NOTE.—To change a *proportion* into an *equation*, it is necessary to assume the truth of the following well established principle:

If four quantities are proportional, the product of the extremes is

* A strict conformity to system would require that this and similar problems should be placed after the subject of proportion; but it is convenient for the learner to be able to convert a proportion into an equation at this stage of his progress.

equal to the product of the means. Hence, in such cases, we have only to make the product of the extremes one side of the equation, and the product of the means the other.

Thus, let $2x$ and $3x$ be equal to their respective ages.

$$\text{Then} \qquad 2x+4 : 3x+4 :: 5 : 7.$$

Making the product of the extremes equal to the product of the means, we have,

$$14x+28 = 15x+20.$$

Transposing, uniting terms, etc., $x = 8$.

$$\therefore\ 2x = 16, \text{ the younger; and } 3x = 24, \text{ the older.}$$

68. What two numbers are as 3 to 4, to each of which if 4 be added, the sums will be as 5 to 6?

69. The sum of two numbers is 5760, and their difference is equal to $\frac{1}{3}$ of the greater. What are the numbers?

70. It takes a college crew which in still water can pull at the rate of 9 miles an hour, twice as long to come up the river as to go down. At what rate does the river flow?

71. One-tenth of a rod is colored red, $\frac{1}{20}$ orange, $\frac{1}{30}$ yellow, $\frac{1}{40}$ green, $\frac{1}{50}$ blue, $\frac{1}{60}$ indigo, and the remainder, 302 inches, violet. What is its length?

72. Of a certain dynasty, $\frac{1}{3}$ of the kings were of the same name, $\frac{1}{4}$ of another, $\frac{1}{8}$ of another, $\frac{1}{12}$ of another, and there were 5 kings besides. How many were there of each name?

73. The difference of the squares of two consecutive numbers is 15. What are the numbers?

74. A deer is 80 of her own leaps before a greyhound; she takes 3 leaps for every 2 that he takes, but he covers as much ground in one leap as she does in two. How many leaps will the deer have taken before she is caught?

75. Two steamers sailing from New York to Liverpool, a distance of 3000 miles, start from the former at the same time, one making a round trip in 20 days, the other in 25 days. How long before they will meet in New York, and how far will each have sailed?

CHAPTER X.

SIMULTANEOUS EQUATIONS.

TWO UNKNOWN QUANTITIES.

218. ***Simultaneous* Equations*** consist of two or more equations, each containing *two or more unknown* quantities, the *respective values* of which are *the same* in each equation.

Thus, $x+y=7$ and $5x-4y=8$ are simultaneous equations, for in each $x=4$ and $y=3$.

219. ***Independent Equations*** are those which express *different conditions*, so that one cannot be reduced to the same form as the other.

Thus, $6x-4y=14$ and $2x+3y=22$ are independent equations. But the equations $x+y=5$ and $3x+3y=15$ are not *independent*, for one is directly obtained from the other. Such equations are termed *dependent*.

NOTE.—*Simultaneous* equations are usually *independent;* but *independent* equations *may* not be *simultaneous;* for the letters employed *may* have the *same* or *different* values in the respective equations.

Thus, the equations $x+y=7$ and $2x-2y=14$ are independent, but not simultaneous; for in one $x=7-y$, in the other $x=7+y$, etc.

220. Problems containing *more than one* unknown quantity must have as *many simultaneous* equations as there are unknown quantities.

If there are *more* equations than unknown quantities, some of them will be *superfluous* or *contradictory*.

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218. What are simultaneous equations? 219. Independent equations? 220. How many equations must each problem have?

* From the Latin *simul*, at the *same time*.

If the number of equations be *less* than the number of unknown quantities, the problem will not admit of a *definite* answer, and is said to be *indeterminate* or *impossible*.

221. ***Elimination**** is combining two equations which contain two unknown quantities into a single equation, having but one unknown quantity. There are three methods of elimination, viz.: by *Comparison*, by *Substitution*, and by *Addition* or *Subtraction*.

CASE I.

222. To Eliminate an Unknown Quantity by *Comparison*.

1. Given $x + y = 16$, and $x - y = 4$, to find x and y.

SOLUTION.—By the problem,	$x+y=16$	(1)
" "	$x-y=4$	(2)
Transposing the y in (1),	$x=16-y$	(3)
" the y in (2),	$x=4+y$	(4)
By Axiom 1,	$4+y=16-y$	(5)
Transposing and uniting terms,	$2y=12$	(6)
	$\therefore\ y=6$	
Substituting the value of y in (4),	$x=10$	

☞ In (5) it will be seen we have a new equation which contains only one unknown quantity. This equation is reduced in the usual way. Hence, the

RULE.—I. *From each equation find the value of the quantity to be eliminated in terms of the other quantities.*

II. *Form a new equation from these equal values, and reduce it by the preceding rules.*

NOTE.—This rule depends upon the axiom, that things which are equal to the same thing are equal to each other. (Ax. 1.)

☞ For convenience of reference, the equations are numbered (1), (2), (3), (4), etc.

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221. What is elimination? Name the methods. 222. How eliminate an unknown quantity by comparison? *Note.* Upon what principle does this rule depend?

* From the Latin *eliminare*, to *cast out*.

2. Given $x + y = 12$, and $x - y + 4 = 8$, to find x and y.
3. Given $3x + 2y = 48$, and $2x - 3y = 6$, to find x and y.
4. Given $x + y = 20$, and $2x + 3y = 42$, to find x and y.
5. Given $4x + 3y = 13$, and $3x + 2y = 9$, to find x and y.
6. Given $3x + 2y = 118$, and $x + 5y = 191$, to find x and y.
7. Given $4x + 5y = 22$, and $7x + 3y = 27$, to find x and y.

CASE II.

223. To Eliminate an Unknown Quantity by *Substitution*.

8. Given $x + 2y = 10$, and $3x + 2y = 18$, to find x and y.

SOLUTION.—By the problem, $x + 2y = 10$ (1)

" " $3x + 2y = 18$ (2)

Transposing $2y$ in (1), $x = 10 - 2y$ (3)

Substituting the value of x in (2), $30 - 4y = 18$ (4)

Transposing and uniting terms (Art. 211), $4y = 12$ (5)

$\therefore y = 3$

Substituting the value of y in (1), $x = 4$

☞ For convenience, we first find the value of the letter which is least involved. Hence, the

RULE.—I. *From one of the equations find the value of the unknown quantity to be eliminated, in terms of the other quantities.*

II. *Substitute this value for the same quantity in the other equation, and reduce it as before.*

NOTES.—1. This method of elimination depends on Ax. 1.

2. The given equations should be cleared of fractions before commencing the elimination.

9. Given $x + 3y = 19$, and $5x - 2y = 10$, to find x and y.
10. Given $\frac{x}{2} + \frac{y}{3} = 7$, and $\frac{x}{3} + \frac{y}{2} = 8$, to find x and y.
11. Given $2x + 3y = 28$, and $3x + 2y = 27$, to find x and y.
12. Given $4x + y = 43$, and $5x + 2y = 56$, to find x and y.
13. Given $5x + 8 = 7y$, and $5y + 32 = 7x$, to find x and y.
14. Given $4x + 5y = 22$, and $7x + 3y = 27$, to find x and y.

223. How eliminate an unknown quantity by substitution? *Note.* Upon what principle does this method depend?

CASE III.

224. To Eliminate an Unknown Quantity by *Addition or Subtraction.*

15. Given $4x+3y=18$, and $5x-2y=11$, to find x and y.

SOLUTION.—By the problem,	$4x+3y=18$	(1)
" "	$5x-2y=11$	(2)
Multiplying (1) by 2, the coef. of y in (2),	$8x+6y=36$	(3)
Multiplying (2) by 3, the coef. of y in (1),	$15x-6y=33$	(4)
Adding (3) and (4) cancels $6y$,	$23x \quad =69$	(5)
	$\therefore x \quad = 3$	
Substituting the value of x in (1),	$12+3y=18$	
	$y=2$	

☞ In the preceding solution, y is eliminated by *addition*.

16. Given $6x+5y=28$, and $8x+3y=30$, to find x and y.

SOLUTION.—By the problem,	$6x+5y=28$	(1)
" "	$8x+3y=30$	(2)
Multiplying (1) by 8, the coef. of x in (2),	$48x+40y=224$	(3)
Multiplying (2) by 6, the coef. of x in (1),	$48x+18y=180$	(4)
Subtracting (4) from (3),	$22y=44$	
	$\therefore y=2$	
Substituting the value of y in (2)	$8x+6=30$	
	$\therefore x=3$	

☞ In this solution, x is eliminated by *subtraction*. Hence, the

RULE.—I. *Select the letter to be eliminated; then multiply or divide one or both equations by such a number as will make the coefficients of this letter the same in both.* (Ax. 4, 5.)

II. *If the signs of these coefficients are alike, subtract one equation from the other; if unlike, add the two equations together.* (Ax. 2, 3.)

NOTES.—1. The *object* of *multiplying* or *dividing* the equations is to *equalize* the *coefficients* of the letter to be eliminated.

2. If the coefficients of the letter to be eliminated are prime numbers, or *prime* to each other, multiply each equation by the coefficient of this letter in the other equation, as in Ex. 15.

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224. What is the rule for elimination by addition or subtraction? *Notes.*—1. The object of multiplying or dividing the equation? 2. If coefficients are prime?

3. If not prime, divide the *l. c. m.* of the coefficients of the letter to be eliminated by each of these coefficients, and the respective quotients will be the multipliers of the corresponding equations. Thus, the *l. c. m.* of 6 and 8, the coefficients of x in Ex. 16, is 24; hence, the multipliers would be 3 and 4.

4. If the coefficients of the letter to be eliminated have *common factors*, the operation is shortened by *cancelling* these factors before the multiplication is performed. Thus, by cancelling the common factor 2 from 6 and 8, the coefficients of x in the last example, they become 3 and 4, and the labor of finding the *l. c. m.* is avoided.

17. Given $3x+4y=29$, and $7x+11y=76$, to find x and y.
18. Given $9x-4y=8$, and $13x+7y=101$, to find x and y.
19. Given $3x-7y=7$, and $12x+5y=94$, to find x and y.
20. Given $3x+2y=118$, and $x+5y=191$, to find x and y.
21. Given $4x+5y=22$, and $7x+3y=27$, to find x and y.

NOTE.—The preceding methods of elimination are applicable to all simultaneous simple equations containing two unknown quantities, and either may be employed at the pleasure of the learner.

The *first method* has the merit of *clearness*, but often gives rise to *fractions*.

The *second* is *convenient* when the coefficient of one of the unknown quantities is 1; if more than 1, it is liable to produce *fractions*.

The *third* never gives rise to fractions, and, in general, is the most simple and expeditious.

EXAMPLES.

Find the values of x and y in the following equations:

1. $2x + 3y = 23$,
   $5x - 2y = 10$.

2. $4x + y = 34$,
   $4y + x = 16$.

3. $30x + 40y = 270$,
   $50x + 30y = 340$.

4. $2x + 7y = 34$,
   $5x + 9y = 51$.

5. $5x + 7y = 43$,
   $11x + 9y = 69$.

6. $8x - 21y = 33$,
   $6x + 35y = 177$.

7. $21y + 20x = 165$,
   $77y - 30x = 295$.

8. $11x - 10y = 14$,
   $5x + 7y = 41$.

Notes.—3. If not prime, how proceed? 4. If the coefficients have common factors, how shorten the operation?

9. $6y - 2x = 208$,
$10x - 4y = 156$.

10. $4x + 3y = 22$,
$5x - 7y = 6$.

11. $3x - 5y = 13$,
$2x + 7y = 81$.

12. $5x - 7y = 33$,
$11x + 12y = 100$.

13. $\frac{x}{5} + \frac{y}{6} = 18$,
$\frac{x}{2} - \frac{y}{4} = 21$.

14. $16x + 17y = 500$,
$17x - 3y = 110$.

15. $8x + y = 42$,
$2x + 4y = 18$.

16. $2x + 4y = 20$,
$4x + 5y = 28$.

17. $4x + 3y = 50$,
$3x - 3y = 6$.

18. $3x + 5y = 57$,
$5x + 3y = 47$.

19. $\frac{x}{2} + \frac{y}{3} = 7$,
$\frac{x}{3} + \frac{y}{4} = 5$.

20. $2x + y = 50$,
$\frac{x}{6} + \frac{y}{7} = 5$.

PROBLEMS.

1. Required two numbers whose sum is 70, and whose difference is 16.

2. A boy buys 8 lemons and 4 oranges for 56 cents; and afterwards 3 lemons and 8 oranges for 60 cents. What did he pay for each?

3. At a certain election, 375 persons voted for two candidates, and the candidate chosen had a majority of 91. How many voted for each?

4. Divide the number 75 into two such parts that three times the greater may exceed seven times the less by 15.

5. A farmer sells nine horses and seven cows for \$1200; and six horses and thirteen cows for an equal amount. What was the price of each?

6. From a company of ladies and gentlemen, 15 ladies retire; there are then left two gentlemen to each lady. After which 45 gentlemen depart, when there are left five ladies to each gentleman. How many were there of each at first?

7. Find two numbers, such that the sum of five times the first and twice the second is 19; and the difference between seven times the first and six times the second is 9.

8. Two opposing armies number together 21,110 men; and twice the number of the greater army added to three times that of the less is 52,219. How many men in each army?

9. A certain number is expressed by two digits. The sum of these digits is 11; and if 13 be added to the first digit, the sum will be three times the second. What is the number?

10. A and B possess together $570. If A's share were three times and B's five times as great as each really is, then both would have $2350. How much has each?

11. If 1 be added to the numerator of a fraction, its value is $\frac{1}{3}$; and if 1 be added to the denominator, its value is $\frac{1}{4}$. What is the fraction.

12. A owes $1200; B, $2550. But neither has enough to pay his debts. Said A to B, Lend me $\frac{1}{8}$ of your money, and I shall be enabled to pay my debts. B answered, I can discharge my debts, if you lend me $\frac{1}{6}$ of yours. What sum has each?

13. Find two numbers whose difference is 14, and whose sum is 48.

14. A house and garden cost $8500, and the price of the garden is $\frac{5}{12}$ the price of the house. Find the price of each.

15. Divide 50 into two such parts that $\frac{3}{4}$ of one part, added to $\frac{5}{6}$ of the other, shall be 40.

16. Divide $1280 between A and B, so that seven times A's share shall equal nine times B's share.

17. The ages of two men differ by 10 years; 15 years ago, the elder was twice as old as the younger. Find the age of each.

18. A man owns two horses and a saddle. If the saddle, worth $50, be put on the first horse, the value of the two is double that of the second horse; but if the saddle be put

on the second horse, the value of the two is \$15 less than that of the first horse. Required the value of each horse.

19. A war-steamer in chase of a ship 20 miles distant, goes 8 miles while the ship sails 7. How far will each go before the steamer overtakes the ship?

20. There are two numbers, such that $\frac{1}{2}$ the greater added to $\frac{1}{3}$ the less is 13; and if $\frac{1}{2}$ the less be taken from $\frac{1}{3}$ the greater, the remainder is nothing. Find the numbers.

21. The mast of a ship is broken in a gale. One-third of the part left, added to $\frac{1}{6}$ of the part carried away, equals 28 feet; and five times the former part diminished by 6 times the latter equals 12 feet. What was the height of the mast?

22. A lady writes a poem of half as many verses less two as she is years old; and if to the number of her years that of her verses be added, the sum is 43. How old is she? How many verses in the poem?

23. What numbers are those whose difference is 20, and the quotient of the greater divided by the less is 3?

24. A man buys oxen at \$65 and colts at \$25 per head, and spends \$720; if he had bought as many oxen as colts, and *vice versa,* he would have spent \$1440. How many of each did he purchase?

25. There is a certain number, to the sum of whose digits if you add 7, the result will be 3 times the left-hand digit; and if from the number itself you subtract 18, the digits will be inverted. Find the number.

26. A and B have jointly \$9800. A invests the sixth part of his property in business, and B the fifth part of his, and each has then the same sum remaining. What is the entire capital of each?

27. A purse holds six guineas and nineteen silver dollars. Now five guineas and four dollars fill $\frac{17}{63}$ of it. How many will it hold of each?

28. The sum of two numbers is a, and the greater is n times the less. What are the numbers?

THREE OR MORE UNKNOWN QUANTITIES.

225. The preceding methods of elimination of two unknown quantities are applicable to equations containing *three* or *more* unknown quantities. (Arts. 222–224.)

226. To Solve Equations containing three or more Unknown Quantities.

1. Given $3x + 2y - 5z = 8$, $2x + 3y + 4z = 16$, and $5x - 6y + 3z = 6$, to find x, y, and z.

SOLUTION.—By the problem,	$3x + 2y - 5z = 8$	(1)
" "	$2x + 3y + 4z = 16$	(2)
" "	$5x - 6y + 3z = 6$	(3)
Multiplying (1) by 2,	$6x + 4y - 10z = 16$	(4)
" (2) by 3,	$6x + 9y + 12z = 48$	(5)
Subtracting (4) from (5),	$5y + 22z = 32$	(6)
Multiplying (2) by 5,	$10x + 15y + 20z = 80$	(7)
" (3) by 2,	$10x - 12y + 6z = 12$	(8)
Subtracting (8) from (7),	$27y + 14z = 68$	(9)
Multiplying (6) by 27,	$135y + 594z = 864$	(10)
" (9) by 5,	$135y + 70z = 340$	(11)
Subtracting (11) from (10),	$524z = 524$	(12)
	$\therefore z = 1$	
Substituting the value of z in (6),	$y = 2$	
Substituting the value of y and z in (2),	$x = 3$	

Ans. $x = 3$, $y = 2$, $z = 1$. Hence, the

RULE.—I. *From the given equations eliminate one unknown quantity, by combining one equation with another.*

II. *From the resulting equations eliminate another unknown quantity in a similar manner. Continue the operation until a single equation is obtained, with but one unknown quantity, and reduce this by the preceding rules.*

NOTE.—The letter having the *smallest* coefficients should be eliminated *first*; and if each letter is not found in all the given equations, begin with that which is in the *least number* of the equations.

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226. What is the rule for solving equations having three or more unknown quantities? *Note.* Which letter should be eliminated first?

Reduce the following equations:

2. $5x - 3y + 2z = 28,$
$3x + 2y - 4z = 15,$
$3y + 4z - x = 24.$

3. $2x + 5y - 3z = 4,$
$4x - 3y + 2z = 9,$
$5x + 6y - 2z = 18.$

4. $2x + 3y - 4z = 20,$
$x - 2y + 3z = 6,$
$3x - 2y + 5z = 26.$

5. $5x + 2y + 4z = 46,$
$3x + 2y + z = 23,$
$10x + 5y + 4z = 75.$

6. $x + y + z = 53,$
$x + 2y + 3z = 105,$
$x + 3y + 4z = 134.$

7. $3x + 4z = 57,$
$2y - z = 11,$
$5x + 3y = 65.$

8. $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 62,\ \frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 47,\ \frac{x}{4} + \frac{y}{5} + \frac{z}{6} = 38.$

9. Required the value of w, x, y, and z in the following equations:

$$
\begin{aligned}
w + x + y + z &= 14 \quad (1)\\
2w + x + y - z &= 6 \quad (2)\\
2w + 3x - y + z &= 14 \quad (3)\\
w - x + 3y + 4z &= 31 \quad (4)
\end{aligned}
$$

SOLUTION.

Adding (1) and (2),	$3w + 2x + 2y$	$= 20$	(5)
" (2) and (3),	$4w + 4x$	$= 20$	(6)
Multiply (3) by 4,	$8w + 12x - 4y + 4z$	$= 56$	(7)
Subtract (4) from (7),	$7w + 13x - 7y$	$= 25$	(8)
Multiply (5) by 7,	$21w + 14x + 14y$	$= 140$	(9)
" (8) by 2,	$14w + 26x - 14y$	$= 50$	(10)
Add (9) and (10),	$35w + 40x$	$= 190$	(11)
Multiply (6) by 10,	$40w + 40x$	$= 200$	(12)
Subtract (11) from (12),	$5w$	$= 10$	
	$\therefore w$	$= 2$	
Substituting value of w in (6),	$8 + 4x$	$= 20$	
	$\therefore x$	$= 3$	

Substituting value of w and x in (5), etc.,

$$y = 4, \text{ and } z = 5.$$

227. The solution of equations containing many unknown quantities may often be shortened by substituting a single letter for several.

10. Required the value of w, x, y, and z in the adjoining equations.

$$\begin{cases} w+x+y=13 & (1) \\ w+x+z=17 & (2) \\ w+y+z=18 & (3) \\ x+y+z=21 & (4) \end{cases}$$

NOTE.—Substituting s for the sum of the four quantities, we have,

$$s=w+x+y+z.$$

Equation (1) contains all the letters but			z,	$s-z=13$	(5)
" (2)	"	" "	y,	$s-y=17$	(6)
" (3)	"	" "	x,	$s-x=18$	(7)
" (4)	"	" "	w,	$s-w=21$	(8)

Adding the last four equations together,

$$\left.\begin{array}{l} 4s-z-y-x-w \\ \text{or } 4s-(z+y+x+w) \\ \text{or } 4s-s \end{array}\right\} = 69 \qquad (9)$$

That is, $3s=69$ (10)

$\therefore\ s=23$

Substituting 23 for s in each of the four equations, we have,

$$w=2, \quad x=5, \quad y=6, \quad z=10.$$

11. Required the value of v, w, x, y, and z, in the adjoining equations.

$$\begin{cases} v+w+x+y=10 & (1) \\ v+w+x+z=11 & (2) \\ v+w+y+z=12 & (3) \\ v+x+y+z=13 & (4) \\ w+x+y+z=14 & (5) \end{cases}$$

NOTE.—Adding these equations, $4v+4w+4x+4y+4z=60$ (6)

Dividing (6) by 4, $v+w+x+y+z=15$ (7)

Subtracting each equation from (7), we have,

$$z=5, \quad y=4, \quad x=3, \quad w=2, \quad \text{and} \quad v=1.$$

12. $w+x+z=10$,
$x+y+z=12$,
$w+x+y=9$,
$w+y+z=11$.

13. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{6}$,
$\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{7}{12}$,
$\dfrac{1}{x}+\dfrac{1}{z}=\dfrac{3}{4}$.

---

227. How shorten the solution of problems containing many unknown quantities?

PROBLEMS.

1. A man has 3 sons; the sum of the ages of the first and second is 27, that of the first and third is 29, and of the second and third is 32. What is the age of each?

2. A butcher bought of one man 7 calves and 13 sheep for $205; of a second, 14 calves and 5 lambs for $300; and of a third, 12 sheep and 20 lambs for $140, at the same rates. What was the price of each?

3. The sum of the first and second of three numbers is 13, that of the first and third 16, and that of the second and third 19. What are the numbers?

4. In three battalions there are 1905 men: $\frac{1}{2}$ the first with $\frac{1}{3}$ in the second, is 60 less than in the third; $\frac{1}{3}$ the the third with $\frac{1}{2}$ the first, is 165 less than the second. How many are in each?

5. A grocer has three kinds of tea: 12 lbs. of the first, 13 lbs. of the second, and 14 lbs. of the third are together worth $25; 10 lbs. of the first, 17 lbs. of the second, and 11 lbs. of the third are together worth $24; 6 lbs. of the first, 12 lbs. of the second, and 6 lbs. of the third are together worth $15. What is the value of a pound of each?

6. Two pipes, A and B, will fill a cistern in 70 minutes, A and C will fill it in 84 minutes, and B and C in 140 min. How long will it take each to fill the cistern?

7. Divide $90 into 4 such parts, that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, shall all be equal.

8. The sum of the distances which A, B, and C have traveled is 62 miles; A's distance is equal to 4 times C's, added to twice B's; and twice A's added to 3 times B's, is equal to 17 times C's. What are the respective distances?

9. A, B, and C purchase a horse for $100. The payment would require the whole of A's money, with half of B's; or the whole of B's with $\frac{1}{3}$ of C's; or the whole of C's with $\frac{1}{4}$ of A's. How much money has each?

CHAPTER XI.

GENERALIZATION.

228. ***Generalization*** is the process of finding a *formula*, or *general rule*, by which all the problems of a *class* may be solved.

229. A ***Problem is generalized*** when stated in general terms which embrace all examples of its class.

230. In all ***General Problems*** the quantities are expressed by *letters*.

1. A marketman has 75 turkeys; if his turkeys are multiplied by the number of his chickens, the result is 225. How many chickens has he?

NOTE.—This problem may be stated in the following general terms:

231. The Product of two *Factors* and one of the Factors being given, to Find the other Factor.*

SUGGESTION.—If a *product* of *two* factors is divided by *one* of them, it is evident the *quotient* must be the *other* factor. Hence, substituting a for the product, b for the given factor, we have the following

GENERAL SOLUTION.—Let $x =$ the required factor.

By the conditions, $x \times b$, or $bx = a$, the product. Hence, the

$$\text{FORMULA.} \qquad x = \frac{a}{b}.$$

Translating this Formula into common language, we have the following

RULE.—*Divide the product by the given factor; the quotient is the factor required.*

---

228. What is generalization? 229. When is a problem generalized? 230. How are quantities expressed in general problems?

* New Practical Arithmetic, Article 93.

Generalize the next two problems:

2. A rectangular field contains 480 square rods, and the length of one side is 16 rods. What is the length of the other side?

3. Divide 576 into two such factors that one shall be 48.

4. The product of A, B, and C's ages is 61,320 years; A is 30 years, B 40. What is the age of C?

NOTE.—The items here given may be generalized as follows:

232. The Product of three *Factors* and two of them being given, to Find the other Factor.

SUGGESTION.—Substituting a for the product, b for one factor, and c for the other, we have the

GENERAL SOLUTION.—Let $x =$ the required factor.

By the conditions, $x \times b \times c$, or $bcx = a$, the product.

Removing the coefficient, we have the

FORMULA. $$x = \frac{a}{bc}.$$

RULE.—*Divide the given product by the product of the given factors; the quotient is the required factor.*

5. The contents of a rectangular block of marble are 504 cubic feet; its length is 9 feet, and its breadth 8 feet. What is its height?

6. The product of 3 numbers is 62,730, and two of its factors are 41 and 45. Required the other factor.

7. The amount paid for two horses was $392, and the difference in their prices was $18. What was the price of each?

NOTE.—From the items given, this problem may be generalized as follows:

---

231. When the product of two factors and one of the factors are given, how find the other factor? 232. When the product of three factors and two of them are given, how find the other factor?

233. The Sum and Difference of two Quantities being given, to Find the Quantities.

SUGGESTION.—Since the *sum* of two quantities equals the greater *plus* the less; and the less *plus* the difference equals the greater; it follows that the *sum plus* the *difference* equals *twice* the greater. Substituting s for the sum, d for the difference, g for the greater, and l for the less, we have the following

GENERAL SOLUTION.—Let $g =$ the greater number,
and $l =$ " less "

Adding, $g+l = s$, the sum.
Subtracting, $g-l = d$, the difference.
Adding sum and difference, $2g = s+d$
Removing coefficient, $g = \frac{s+d}{2}$, *greater.*
Subtracting difference from sum, $2l = s-d$
Removing coefficient, $l = \frac{s-d}{2}$, *less.* Hence, the

FORMULAS. $$\left\{ \begin{array}{l} g = \frac{s+d}{2}. \\ l = \frac{s-d}{2}. \end{array} \right\}$$

☞ This problem may be solved by *one* unknown quantity.

FORMATION OF RULES.

234. Many of the more important rules of Arithmetic are formed by translating *Algebraic Formulas* into common language. Thus, from the translation of the two preceding formulas into common language, we have, for all problems of this class, the following general

RULE.—I. *To find the greater, add half the sum to half the difference.*

II. *To find the less, subtract half the difference from half the sum.*

8. Divide $1575 between A and B in such a manner that A may have $347 more than B. What will each receive?

233. When the sum and difference of two quantities are given, how find the quantities? 234. Give the rule derived from the last two formulas.

9. At an election there were 2150 votes cast for two persons; the majority of the successful candidate was 346. How many votes did each receive?

10. If B can do a piece of work in 8 days, and C in 12 days, how long will it take both to do it?

NOTE.—Regarding the work to be done as a *unit* or 1, the problem may be thus generalized:

235. The Time being given in which each of two or more Forces can produce a given Result, to Find the Time required by the united Forces to produce it.

SUGGESTION.—Since B can do the work in 8 days, he can do 1 eighth of it in 1 day, and C can do 1 twelfth of it in 1 day. Substituting a for 8 days, and b for 12 days, we have the

GENERAL SOLUTION.—Let $x =$ the time required.

Dividing x by a, we have $\frac{x}{a} =$ part done by B.

" x by b, we have, $\frac{x}{b} =$ " " C.

By Axiom 9, $\frac{x}{a} + \frac{x}{b} = 1$, the work done.

Clearing of fractions, $bx + ax = ab$

Uniting the terms, $(a+b)x = ab$, B and C's time.

Removing the coefficient, we have the

FORMULA. $$x = \frac{ab}{a+b}.$$

RULE.—*Divide the product of the numbers denoting the time required by each force, by the sum of these numbers; the quotient is the time required by the united forces.*

11. A cistern has two pipes; the first will fill it in 9 hours, the second in 15 hours. In what time will both fill it, running together?

---

235. The time being given in which two or more forces can produce a result, how find the time required for the united forces to produce it?

239. The Percentage and Rate being given, to Find the *Base*.

SUGGESTION.—We have the *product* and *one* of its factors given, to find the *other* factor. (Art. 237, *note*.) Substituting p for the *percentage*, and r for the *rate per cent* gained, we have the

GENERAL SOLUTION.—Let $b=$ the base.

Then (Art. 237), $p \div r = b$. Hence, the

FORMULA. $$b = \frac{p}{r}.$$

RULE.—*Divide the percentage by the rate, and the quotient is the base.*

21. A paid a tax of \$750, which was 2 per cent of his property. How much was he worth?

22. A merchant saves 8 per cent of his net income, and lays up \$2500 a year. What is his income?

23. At the commencement of business, B and C were each worth \$2500. The first year B added 8 per cent to his capital, and C lost 8 per cent of his. What amount was each then worth?

NOTE.—The items of this problem may be generalized thus:

240. The Base and Rate being given, to Find the *Amount*.

SUGGESTION.—Since B laid up 8 per cent., he was worth his original stock *plus* 8 per cent of it. But his stock was 100 per cent, or *once* itself; and 100 per cent. + .08 = 108 per cent or 1.08 times his stock.

Again, since C lost 8 per cent, he was worth his original stock *minus* 8 per cent of it. Now 100 per cent *minus* 8 per cent equals 100 per cent — .08 = 92 per cent, or .92 times his capital. Substituting b for the base or capital of each, and r for the number denoting the *rate per cent* of the gain or loss, we have the

GENERAL SOLUTION.—Let $a =$ the amount.

Then will $b(1+r) = a$, B's amount.

And $b(1-r) = a$, C's amount.

Combining these two results, we have the

FORMULA. $$a = b\,(1 \pm r).$$

RULE.—*Multiply the base by* 1 ± *the rate, as the case may require, and the result will be the amount.*

---

239. When percentage and rate are given, how find the base? 240. How find the amount when the base and rate are given?

NOTE.—When, from the nature of the problem, the *amount* is to be *greater* than the base, the multiplier is 1 *plus* the rate; when *less*, the multiplier is 1 *minus* the rate.

24. A man bought a flock of sheep for $4500, and sold it 25 per cent above the cost. What amount did he get for it?

25. A man owned 2750 acres of land, and sold 33 per cent of it. What amount did he have left?

241. The elements or factors which enter into computations of interest are the *principal, rate, time, interest,* and *amount.* Thus,

Let p = the principal, or money lent.
" r = the interest of $1 for 1 year, at the given rate.
" t = the time in years.
" i = the interest, or the percentage.
" a = the amount, or the *sum* of principal and interest.

26. What is the interest of $465 for 2 years, at 6 per cent?

NOTE.—The data of this problem may be stated in the following general proposition:

242. The Principal, the Rate, and Time being given, to Find the *Interest.*

GENERAL SOLUTION.—Since r is the interest of $1 for 1 year, $p \times r$ must be the interest of p dollars for 1 year; therefore, $pr \times t$ must be the interest of p dollars for t years. Hence, the

FORMULA. $i = prt.$

RULE.—*Multiply the principal by the interest of $1 for the given time, and the result is the interest.*

27. What is the interest of $1586 for 1 yr. and 6 m., at 8 per cent?

28. What is the int. of $3580 for 5 years, at 7 per cent?

29. What is the amt. of $364 for 3 years, at 5 per cent?

NOTE.—This problem may be stated in the following general terms:

Note. When the amount is greater or less than the base, what is the multiplier?
241. What are the elements or factors which enter into computations of interest?
242. When the principal, rate, and time are given, how find the interest?

243. The Principal, Rate, and Time being given, to Find the *Amount.*

GENERAL SOLUTION.—Reasoning as in the preceding article,

the interest $= prt$.

But the *amount* is the *sum* of the principal and interest.

$\therefore \quad p+prt = a.$ Hence, the

FORMULA. $$a = p + prt.$$

RULE.—*Add the interest to the principal, and the result is the amount.*

30. Find the amount of $4375 for 2 years and 6 months, at 8 per cent.

31. Find the amt. of $2863.60 for 5 years, at 7 per cent.

244. The ***Relation*** between the *four elements* in the

FORMULA, $$a = p + prt,$$

is such, that if any *three* of them are given, the *fourth* may be readily found. (Art. 243.)

245. The Amount, the Rate, and Time being given, to Find the *Principal.*

Transposing the members and factoring, we have the

FORMULA. $$p = \frac{a}{1 + rt}.$$

32. What principal will amount to $1500 in 2 years, at 6 per cent?

33. What sum must be invested at 7 per cent to amount to $300 in 5 years?

246. The Amount, the Principal, and the Rate being given, to Find the *Time.*

Transposing p and dividing by pr, (Art. 244), we have the

FORMULA. $$t = \frac{a - p}{pr}.$$

34. In what time will $3500, at 6 per cent, yield $525 interest?

---

243. The amount? 244. What is the relation between the four elements in the preceding formula. 245. When the amount, rate, and time are given, state the formula. 246. When the amount, principal, and rate are given, state the formula.

247. The Hour and Minute Hands of a Clock being together at 12 M., to Find the Time of their *Conjunction* between any two *Subsequent* Hours.

35. The hour and minute hands of a clock are exactly together at 12 o'clock. It is required to find how long before they will be together again.

ANALYSIS.—The distance around the dial of a clock is 12 hour spaces. When the hour-hand arrives at I, the minute-hand has passed 12 hour spaces, and made an entire circuit. But since the hour-hand has moved over one space, the minute-hand has gained only 11 spaces. Now, if it takes the minute-hand 1 hour to gain 11 spaces, to gain 1 space will take $\frac{1}{11}$ of an hour, and to gain 12 spaces it will take 12 times as long, and 12 times $\frac{1}{11}$ hr. $= \frac{12}{11}$ hr. $= 1\frac{1}{11}$ hour. Or,

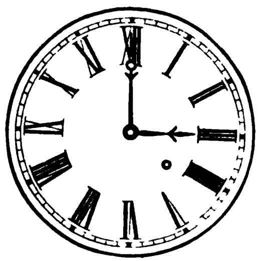
Let $x =$ the time of their conjunction.

Then 11 spaces : 12 spaces :: 1 hour : x hours.

Multiplying extremes, etc., $11x = 12$

Removing coefficient, $x = 1\frac{1}{11}$ hr., or 1 hr. $5\frac{5}{11}$ min.

36. When will the hour and minute hand be in conjunction next after 3 o'clock?



SUGGESTION.—Substituting a for $1\frac{1}{11}$ hr., the time it takes the minute-hand to gain 12 spaces, h for the given number of hours past 12 o'clock, t for the time of conjunction, we have the following

GENERAL SOLUTION. $a \times h = t$, the time required. Hence, the

FORMULA. $t = ah.$

RULE.—*Multiply the time required to gain* 12 *spaces by the given hour past* 12 *o'clock; the product will be the time of conjunction.*

37. At what time after 6 o'clock will the hour and minute hand be in conjunction?

38. At what time between 9 and 10 o'clock will the hour and minute hand be in conjunction?

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247. What is the formula for finding when the hands of a clock will be in conjunction? Translate this into a rule?

CHAPTER XII.

INVOLUTION.*

248. ***Involution*** is finding a power of a quantity.

249. A ***Power*** is the product of two or more equal factors.

Thus, $3 \times 3 = 9$; $a \times a \times a = a^3$; **9 and a^3 are powers.**

250. Powers are divided into *different degrees;* as *first, second, third, fourth,* etc., the name corresponding with the *number of times* the quantity is taken as a *factor* to produce the power.

251. The ***First Power*** is the *quantity* itself.

The ***Second Power*** is the product of a quantity taken *twice* as a factor, and is called a *square.*

The ***Third Power*** is the product of a quantity taken *three times* as a factor, and is called a *cube,* etc.

NOTE.—The quantity *called* the *first power* is, strictly speaking, not a power, but a *root.* Thus, a^1 or a, is not the product of any *two equal* factors, but is a quantity or *root* from which its *powers* arise.

252. The ***Index*** or ***Exponent*** † of a power is a *figure* or *letter* placed at the right, above the quantity. Its object is to show *how many times* the quantity is taken as a *factor* to produce the power.

Thus, $a^1 = a$, and is called the *first* power.

$a^2 = a \times a$, the *second* power, or *square.*

$a^3 = a \times a \times a$, the *third* power, or *cube.*

$a^4 = a \times a \times a \times a$, the *fourth* power, etc.

---

248. What is involution? 249. A power? 250. How divided? 251. The first power? Second power? Third? 252. What is the index or exponent? Its object?

* *Involution,* from the Latin *involvere,* to roll up.

† *Index* (plural, *indices*), Latin *indicāre,* to *indicate.*
Exponent, from the Latin *exponere,* to *set forth.*

NOTES.—1. The index of the *first* power being 1, is commonly omitted.

2. The expression a^4 is read "*a* fourth," "the fourth power of *a*," or "*a* raised to the fourth power;" x^n is read, "*x* *n*th," or "the *n*th power of *x*."

Read the following: a^5, c^4, x^7, y^{10}, z^{15}, b^m, d^n.

253. ***Powers*** are also divided into *direct* and *reciprocal*.

254. ***Direct Powers*** are those which arise from the continued multiplication of a quantity into itself.

Thus, the continued multiplication of a into itself gives the series,

$$a,\ a^2,\ a^3,\ a^4,\ a^5,\ a^6,\ \text{etc.}$$

255. ***Reciprocal Powers*** are those which arise from the continued division of a unit by the *direct powers* of that quantity. (Art. 55.)

Thus, the continued division of a *unit* by the direct powers of a gives the series,

$$\frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \frac{1}{a^4},\ \frac{1}{a^5},\ \frac{1}{a^6},\ \text{etc.}$$

256. ***Reciprocal Powers*** are commonly denoted by prefixing the sign $-$ to the *exponents* of direct powers of the *same* degree.

Thus, $\frac{1}{a}=a^{-1}$, $\frac{1}{a^2}=a^{-2}$, $\frac{1}{a^3}=a^{-3}$, $\frac{1}{a^4}=a^{-4}$, etc.

257. The difference in the notation of *direct* and *reciprocal* powers may be seen from the following series:

(1.) $a^5,\ a^4,\ a^3,\ a^2,\ a^1,\ 1,\ \frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \frac{1}{a^4},\ \frac{1}{a^5}$, etc.

(2.) $a^5,\ a^4,\ a^3,\ a^2,\ a^1,\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4},\ a^{-5}$, etc.

NOTE.—The *first half* of each of the above expressions is a series of *direct* powers; the *last half*, a series of *reciprocal* powers.

258. ***Negative Exponents*** are the same as the exponents of *direct* powers, with the sign $-$ prefixed to them.

Note. The index 1? 253. How else are powers divided? 254. Direct powers? 255. Reciprocal? 256. How is a reciprocal power denoted?

NOTES.—1. This *notation* of reciprocal powers is derived from the *continued division* of a series of *direct* powers by their *root*; that is, by subtracting 1 from the successive exponents. (Art. 113.)

2. The use of negative exponents in expressing reciprocal powers avoids fractions, and therefore is convenient in calculations.

3. *Direct powers* are often called *positive* and *reciprocal* powers *negative*. But the student must not confound the quantities whose *exponents* have the sign + or − with those whose *coefficients* have the sign + or −. This ambiguity will be avoided by applying the term *direct* to powers with *positive* exponents, and *reciprocal* to those with *negative* exponents.

259. The ***Zero Power*** of a quantity is one whose exponent is 0; as, a^0.

Every quantity with the index 0, is equal to a unit or 1.

Thus, $a^0 = 1$, $m^0 = 1$, $x^0 = 1$, etc. (Art. 257.)

SIGNS OF POWERS.

260. When a quantity is *positive*, all its *powers* are *positive*.

Thus, $a \times a = a^2$; $a \times a \times a = a^3$, etc.

When a quantity is *negative*, its *even* powers are *positive*, and its *odd* powers *negative*.

Thus, $-a \times -a = a^2$; $-a \times -a \times -a = -a^3$, etc.

FORMATION OF POWERS.

261. ***All Powers*** of a quantity may be formed by multiplying the quantity into itself. (Art. 249.)

262. To Raise a *Monomial* to any Required Power.

The process of involving a quantity which consists of several factors depends upon the following

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259. What is the zero power? To what is a quantity of the zero power equal?
260. Rule for the signs?

PRINCIPLES.

1°. *The power of the product of two or more factors is equal to the product of their powers.*

2°. *The product is the same, in whatever order the factors are taken.* (Art. 87, Prin. 3.)

1. Given $3ab^2$ to be raised to the third power.

SOLUTION.

$$(3ab^2)^3 = 3ab^2 \times 3ab^2 \times 3ab^2 \text{ (Art. 261)},$$

$$\text{or,} \quad 3 \times 3 \times 3 \times a \times a \times a \times b^2 \times b^2 \times b^2 \text{ (Prin. 2)},$$

$$\therefore \quad (3ab^2)^3 = 27a^3b^6, \textit{ Ans.}$$

Involving each of these factors separately, we have, $(3)^3 = 27$; $(a)^3 = a^3$; and $(b^2)^3 = b^6$; and $27 \times a^3 \times b^6 = 27a^3b^6$, *Ans.* Hence, the

RULE.—*Raise the coefficient to the power required, and multiply the index of each letter by the index of the power, prefixing the proper sign to the result.* (Art. 90.)

NOTES.—1. A *single letter* is involved by giving it the index of the required power.

2. A quantity which is *already a power* is involved by multiplying its *index* by the *index* of the required power.

3. The learner must observe the distinction between an *index* and a *coefficient*. The *latter* is simply a *multiplier*, and shows how many times the quantity is *added* to itself; the *former* shows how many times the quantity is taken as a *factor*. (Arts. 21, 252.)

2. What is the square of abc?
3. What is the square of $-abc$?
4. What is the cube of xyz?
5. What is the fifth power of abc?
6. What is the fourth power of $2x^2y$?
7. What is the third power of $6a^3b^2$?
8. What is the fourth power of $5a^3b^2c$?
9. What is the sixth power of $2a^2bc^2$?
10. What is the eighth power of $abcd$?
11. What is the nth power of xyz?

262. How raise a monomial to any power? *Note.* A single letter? A quantity already a power? Distinction between index and coefficient?

12. Find the fifth power of $(a + b)^2$.
13. Find the second power of $(a + b)^n$.
14. Find the nth power of $(x - y)^m$.
15. Find the nth power of $(x + y)^2$.
16. Find the second power of $(a^3 + b^3)$.
17. Find the third power of $(a^3b^2h^4)$.

263. To involve a *Fraction* to any required Power.

18. What is the square of $\frac{2x^2y}{3x^3y^2}$?

SOLUTION. $\left(\frac{2x^2y}{3x^3y^2}\right)^2 = \frac{2x^2y}{3x^3y^2} \times \frac{2x^2y}{3x^3y^2} = \frac{4x^4y^2}{9x^6y^4}$. Hence, the

RULE.—*Raise both the numerator and denominator to the required power.*

19. Find the cube of $\frac{3ab^2}{2a}$.
20. Find the fourth power of $\frac{2a^2bc^3}{x^2y^3}$
21. Find the square of $\frac{7a^nb^2}{3a^2b^n}$.
22. Find the mth power of $\frac{2}{a}$.
23. Find the nth power of $\frac{a^mb^n}{xy^n}$.

264. A *compound quantity* consisting of two or more terms, connected by + or —, is involved by *actual multiplication* of its several parts.

24. Find the square of $3a + b^2$. *Ans.* $9a^2 + 6ab^2 + b^4$.
25. What is the square of $a + b + c$?
Ans. $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$.
26. What is the cube of $x + 2y + z$?

265. It is sometimes sufficient to express the power of a compound quantity by exponents.

Thus, the square of $a+b = (a+b)^2$; the nth power of $ab+c+3d^2 = (ab+c+3d^2)^n$.

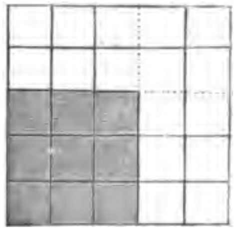
263. How involve a fraction? 264. How involve a compound quantity?

FORMATION OF SQUARES.

266. To Find the *Square* of a *Binomial* in the Terms of its Parts.

1. Given two numbers, 3 and 2, to find the square of their sum in the terms of its parts.

ILLUSTRATION.—Let the shaded part of the diagram represent the square of 3;—each side being divided into 3 inches, its contents are equal to 3×3, or 9 sq. in.



To preserve the form of the square, it is plain equal additions must be made to *two adjacent* sides; for, if made on *one side*, or on *opposite* sides, the figure will no longer be a *square*.

Since 5 is 2 more than 3, it follows that *two rows* of 3 squares each must be added at the top, and 2 rows on one of the adjacent sides, to make its *length* and *breadth* each equal to 5. Now 2 into 3 plus 2 into 3 are 12 squares, or *twice* the product of the two parts 2 and 3.

But the diagram wants two times 2 small squares, to fill the corner on the right, and 2 times 2, or 4, is the square of the second part. We have then 9 (the square of the first part), 12 (twice the product of the two parts 3 and 2), and 4 (the square of the second part). Therefore, $(3+2)^2 = 3^2 + 2 \times (3 \times 2) + 2^2$.

2. Required the square of $x+y$. *Ans.* $x^2 + 2 \times xy + y^2$.

Hence, universally,

The square of the sum of two quantities is equal to the square of the first, plus twice their product, plus the square of the second.

NOTE.—The *square of a binomial* always has *three* terms, and consequently is a *trinomial*. Hence,

No binomial can be a perfect square. (Art. 101.)

266. To what is the square of the sum of two quantities equal? How illustrate the square of the sum of two quantities in the terms of its parts?

267. ***All Binomials*** may be raised to any required power by continued multiplication. But when the exponent of the power is large, the operation is greatly abridged by means of the *Binomial Theorem.**

268. The ***Binomial Theorem*** is a *general formula* by which any power of a binomial may be found without recourse to continued multiplication.

To illustrate this theorem, let us raise the binomials $a+b$ and $a-b$ to the second, third, fourth, and fifth powers, by continued multiplications:

$$(a+b)^2 = a^2 + 2ab + b^2.$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

$$(a-b)^2 = a^2 - 2ab + b^2.$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$
$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$
$$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

269. Analyzing these operations, the learner will discover the following *laws* which govern the *expansion* of *binomials:*

1. *The number of terms* in any power is *one more* than the *index* of the power.

2. The *index* of the *first* term or *leading* letter is the *index* of the required power, which *decreases* regularly by 1 through the other terms.

The *index* of the *following* letter begins with 1 in the second term, and *increases* by 1 through the other terms.

3. *The sum of* the indices is the *same* in each term, and is *equal* to the *index* of the power.

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268. What is the Binomial Theorem? 269. What is the law respecting the number of terms in a power? The indices of each quantity? The sum of the indices in each term?

* This method was invented by Sir Isaac Newton, in 1666.

4. *The coefficient* of the *first* and *last* term of every power is 1; of the *second* and *next* to the last, it is the *index* of the power; and, universally, the coefficients of any two terms equidistant from the extremes, are *equal* to each other.

Again, the coefficients regularly *increase* in the *first half* of the terms, and *decrease* at the same rate in the *last half.*

5. *The signs* follow the *same* rule as in multiplication.

270. The preceding principles may be summed up in the following

GENERAL RULE.

I. INDICES.—*Give the first term or leading letter the index of the required power, and diminish it regularly by* 1 *through the other terms.*

The index of the following letter in the second term is 1, *and increases regularly by* 1 *through the other terms.*

II. COEFFICIENTS.—*The coefficient of the first term is* 1.

To the second term give the index of the power; and, universally, multiplying the coefficient of any term by the index of the leading letter in that term, and dividing the product by the index of the following letter increased by 1, *the result will be the coefficient of the succeeding term.*

III. SIGNS.—*If both terms are positive, make all the terms positive; if the second term is negative, make all the odd terms, counting from the left, positive, and all the even terms negative.*

THE BINOMIAL FORMULA.

$$(a+b)^n = a^n + n \times a^{n-1}b + n \times \frac{n-1}{2}a^{n-2}b^2, \text{ etc.}$$

NOTE.—The preceding rule is based upon the supposition that the index is a *positive whole* number; but it is equally true when the index is either *positive* or *negative, integral* or *fractional.*

The coefficients of the first and last terms? The law of the signs? 270. What is the general rule?

Expand the following binomials:

1. $(a + b)^4$.
2. $(a - b)^5$.
3. $(c + d)^7$.
4. $(x + y)^6$.
5. $(x - y)^7$.
6. $(y + z)^{10}$.
7. $(a - b)^9$.
8. $(m + n)^{11}$.
9. $(x - y)^{12}$.
10. $(a + b)^n$.

271. When the *terms* of a binomial have *coefficients* or *exponents*, the operation may be shortened by substituting for them single letters of the first power. After the operation is completed, the value of the terms must be restored.

11. Required the fifth power of $x^2 + 3y^2$.

SOLUTION.—Substitute a for x^2, and b for $3y^2$; then

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Restoring the values of a and b,

$$(x^2+3y^2)^5 = x^{10} + 15x^8y^2 + 90x^6y^4 + 270x^4y^6 + 405x^2y^8 + 243y^{10}.$$

12. Expand $(x^2 - 3b)^4$.
Ans. $x^8 - 12x^6b + 54x^4b^2 - 108x^2b^3 + 81b^4$.

272. *Every power* of 1 is 1, and when a factor it has no effect upon the quantity with which it is connected. (Art. 94, *note.*) Hence, when one of the terms of a binomial is 1, it is commonly omitted in the required power, except in the *first* and *last* terms.

NOTE.—In finding the exponents of such binomials, it is only necessary to observe that the *sum* of the two exponents in each term is *equal* to the index of the power.

13. Expand $(x + 1)^3$.
14. Expand $(b - 1)^4$.
15. Expand $(1 - a)^5$.
16. Expand $(1 + a)^n$.

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271. When the terms have coefficients or exponents, how may the operation be shortened? 272. When one of the terms of a binomial is 1, what effect has it?

273. A *Polynomial* may be raised to any power by actual multiplication, taking the given quantity as a factor as many times as indicated by the *exponent* of the required power. But the operation may often be shortened by reducing the several terms to two, by substitution, and then applying the Binomial Formula.

17. Required the cube of $x + y + z$.

SOLUTION.—Substituting a for $(y+z)$, we have $x+(y+z) = x+a$.
By formula, $(x+a)^3 = x^3+3x^2a+3xa^2+a^3$.
Restoring the value of a,

$$(x+y+z)^3 = x^3+3x^2(y+z)+3x(y+z)^2+(y+z)^3.$$

274. To *Square* a Polynomial without Recourse to Multiplication.

18. Required the square of $a + b + c$.

SOLUTION.—By actual multiplication, we have,

$$(a+b+c)^2 = a^2+2ab+2ac+b^2+2bc+c^2.$$

Or, changing the order of terms,

$$a^2+b^2+c^2+2ab+2ac+2bc.$$

Or, factoring, we have, $a^2+2a(b+c)+b^2+2bc+c^2$.

19. Required the square of $a + b + c + d$.

SOLUTION.—By actual multiplication, we have,

$$a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

Or, changing the order of the terms, and factoring, we have,
$a^2+2a(b+c+d)+b^2+2b(c+d)+c^2+2cd+d^2$. Hence. the

RULE.—*To the sum of the squares of the terms add twice the product of each pair of terms.*

Or, *To the square of each term add twice its product into the sum of all the terms which follow it.*

20. Required the square of $x + y + z$.
21. Required the square of $a - b + c$.
22. Required the square of $a + x + y + z$.

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273. How may a polynomial be raised to any required power? 274. What is the rule for squaring a polynomial?

275. When one of the terms of a binomial is a *fraction*, it may be involved by actual multiplication, or by reducing the mixed quantity to an improper fraction, and then involving the fraction. (Art. 171.)

23. Required the square of $x + \frac{1}{2}$; and $x - \frac{1}{2}$.

$$\begin{array}{ll} x + \frac{1}{2} \\ \underline{x + \frac{1}{2}} \\ x^2 + \frac{1}{2}x \\ \underline{\quad + \frac{1}{2}x + \frac{1}{4}} \\ x^2 + \quad x + \frac{1}{4} \end{array} \qquad \begin{array}{ll} x - \frac{1}{2} \\ \underline{x - \frac{1}{2}} \\ x^2 - \frac{1}{2}x \\ \underline{\quad - \frac{1}{2}x + \frac{1}{4}} \\ x^2 - \quad x + \frac{1}{4} \end{array}$$

Or, reduce the mixed quantities to improper fractions. Thus,

$$x + \frac{1}{2} = \frac{2x+1}{2}; \quad \text{and} \quad x - \frac{1}{2} = \frac{2x-1}{2}. \quad \text{(Art. 171.)}$$

$$\left(\frac{2x+1}{2}\right)^2 = \frac{4x^2+4x+1}{4}; \quad \text{and} \quad \left(\frac{2x-1}{2}\right)^2 = \frac{4x^2-4x+1}{4}.$$

Expand the following mixed quantities:

24. $(a + \frac{2}{3})^2$.

25. $\left(a - \frac{c}{2}\right)^2$.

26. $(-\frac{6}{7} + 2abc)^2$.

27. $\left(-\frac{b}{m} + 3xy\right)^2$.

276. Powers are ***added and subtracted*** like other quantities. (Arts. 67, 77.) For, the *same* powers of the *same* letters are *like quantities;* while powers of *different* letters and *different* powers of the *same* letter are *unlike* quantities, and are treated accordingly. (Arts. 43, 44.)

28. To $\quad 7a^2 + 5(a+b)^3 - 6x + 3x^3 + a^5$
Add $\quad \underline{-3a^2 + 4(a+b)^3 + 4x + 4x^2 - a^4}$
Ans. $\quad 4a^2 + 9(a+b)^3 - 2x + 3x^3 + 4x^2 + a^5 - a^4$

29. From $\quad 3a^3 + 5b^2 - 4c^3 + 4x^2 - a^5$
Take $\quad \underline{-4a^3 + 3b^2 + 3c^3 - 5x^3 + a^4}$
Ans. $\quad 7a^3 + 2b^2 - 7c^3 + 5x^3 + 4x^2 - a^5 - a^4$

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275. How involve a binomial, when one term is a fraction? 276. How are powers added and subtracted? Why?

MULTIPLICATION OF POWERS.

277. To *Multiply* Powers of the *Same Root.*

1. What is the product of $3a^4b^3$ multiplied by a^2b^2?

SOLUTION.—Adding the exponents of each letter, we have $3a^6$ and b^5. Now $3a^6 \times b^5 = 3a^6b^5$, *Ans.* (Art. 94.)

2. Multiply $3a^5b^4$ by $a^{-3}b^{-2}$.

SOLUTION.—Adding the exponents of each letter, as before, we have $3a^2b^2$, *Ans.* Hence, the

RULE.—*Add the exponents of the given quantities, and the result will be the product.* (Art. 94.)

NOTES.—1. This rule is applicable to *positive* and *negative* exponents.

2. Powers of different roots are multiplied by writing them one after another.

Multiply the following powers:

3. a^6 by a^2.
4. x^{-5} by x^{-8}.
5. b^{-7} by b^4.
6. a^m by a^n.
7. $a^{-4}b$ by $a^{-8}b^4$.
8. $a^{-4}cd$ by $a^7c^8d^2$.
9. $b^4c^{-6}y^{-2}$ by $b^{-2}c^8y^4$.
10. $a^6y^{-4}z^8$ by $a^{-8}y^{-5}z^4$.

DIVISION OF POWERS.

278. To *Divide* Powers of the Same Root.

11. Divide a^5 by a^3.

SOLUTION.—Subtracting one exponent from the other, we have $a^5 \div a^3 = a^2$. the quotient sought. (Art. 113.) Hence, the

RULE.—*Subtract the exponent of the divisor from that of the dividend; the result is the quotient.* (Art. 113.)

NOTE.—This rule is applicable to *positive* and *negative* exponents.

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277. How multiply powers of the same root? *Note.* Of different roots? 278. What is the rule for dividing powers of the same root?

Divide the following powers:

12. a^3 by a^{-3}.
13. x^{-3} by x^3.
14. b^4 by b^2.
15. c^{-5} by c^{-3}.
16. x^4yz^3 by $x^{-3}y^4z^{-3}$.
17. $12a^3b^{-2}c$ by $3a^2b^{-3}c^5$.
18. $6x^4y^2z^3$ by $2x^{-2}yz^2$.
19. $60a^3b^4c^5$ by $5a^{-2}b^4c^{-3}$.

279. The ***Method*** of denoting *Reciprocal Powers* shows that *any factor* may be transferred from the *numerator* of a fraction to the denominator, and *vice versa*, by *changing* the *sign* of its *exponent* from + to −, or − to +.

20. Transfer the denominator of $\frac{a^5}{x^3}$ to the numerator.

SOLUTION. $\frac{a^5}{x^3} = a^5 \times \frac{1}{x^3} = a^5 \times x^{-3} = a^5x^{-3}$, *Ans.*

21. Transfer the denominator of $\frac{a}{x^{-6}}$ to the numerator.

SOLUTION. $\frac{a}{x^{-6}} = \frac{a}{\frac{1}{x^6}} = a \div \frac{1}{x^6} = a \times x^6 = ax^6$, *Ans.*

22. Transfer the numerator of $\frac{a^5}{y}$ to the denominator.

SOLUTION. $\frac{a^5}{y} = \frac{1}{y} \times a^5 = \frac{1}{y} \div \frac{1}{a^5} = \frac{1}{y} \div a^{-5} = \frac{1}{a^{-5}y}$, *Ans.*

23. Transfer the numerator of $\frac{a^{-6}}{y}$ to the denominator.

SOLUTION. $\frac{a^{-6}}{y} = \frac{1}{a^6} \times \frac{1}{y} = \frac{1}{a^6y}$, *Ans.*

24. Transfer x^{-3} to the denominator of $\frac{ax^{-3}}{y}$.

25. Transfer y^4 to the numerator of $\frac{a}{by^4}$.

26. Transfer d^{-5} to the denominator of $\frac{ad^{-5}}{x^2}$.

27. Transfer x^n to the numerator of $\frac{b}{ax^n}$.

279. What inference may be drawn from the method of denoting reciprocal powers? How transfer a factor?

CHAPTER XIII.

EVOLUTION.*

280. ***Evolution*** is finding a *root* of a quantity. It is often called the *Extraction* of roots.

281. A ***Root*** is one of the *equal factors* of a quantity.

NOTES.—1. *Powers* and *roots* are correlative terms. If one quantity is a *power* of another, the latter is a *root* of the former.

Thus, a^3 is the cube of a, and a is the cube root of a^3.

2. The learner should observe the following distinctions:

1st. By *involution* a product of *equal factors* is found.

2d. By *evolution* a quantity is resolved into *equal factors*. It is the reverse of involution.

3d. By *division* a quantity is resolved into *two factors*.

4th By *subtraction* a quantity is separated into *two parts*.

282. *Roots*, like powers, are divided into *degrees;* as, the square, or second root; the cube, or third root; the fourth root, etc.

283. The ***Square Root*** is one of the *two equal factors* of a quantity.

Thus, $5 \times 5 = 25$, and $a \times a = a^2$; therefore 5 is the square root of 25, and a the square root of a^2.

284. The ***Cube Root*** is one of the three equal factors of a quantity.

Thus, $3 \times 3 \times 3 = 27$, and $a \times a \times a = a^3$; therefore, 3 is the cube root of 27, and a is the cube root of a^3.

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280. What is evolution? 281. A root? *Note.* Of what is evolution the reverse? 283. What is the square root? 284. Cube root?

* From the Latin *evolvere*, to unfold.

285. Roots are denoted in two ways:

1st. By prefixing the *radical* sign ($\sqrt{}$) to the quantity.*

2d. By placing a *fractional* exponent on the right of the quantity.

Thus, $\sqrt[2]{a}$ and $a^{\frac{1}{2}}$ denote the square root of a.

$\sqrt[3]{a}$ and $a^{\frac{1}{3}}$ denote the cube root of a, etc.

NOTES.—1. The *index*, or figure placed over the radical sign, is called the *index* of the root because it denotes the *name* of the root.

Thus, $\sqrt[2]{a}$, and $\sqrt[3]{a}$, denote the square and cube root of a.

2. In expressing the *square* root, it is customary to use simply the radical sign ($\sqrt{}$), the 2 being understood.

Thus, the expression $\sqrt{25} = 5$, is read, "the square root of $25 = 5$."

3. The method of expressing roots by *fractional exponents* is derived from the manner of denoting powers by *integral* indices.

Thus, $a^4 = a \times a \times a \times a$; hence, if a^4 is divided into *four equal* factors, *one* of these equal factors may properly be expressed by $a^{\frac{1}{4}}$.

286. The *numerator* of a fractional exponent denotes the *power*, and the *denominator* the root.

Thus, $a^{\frac{1}{3}}$ denotes the *cube* root of the *first* power of a; and $a^{\frac{3}{4}}$ denotes the *fourth* root of the *third* power of a, or the *third* power of the *fourth* root, etc.

Read the following expressions:

1. $a^{\frac{3}{4}}$.	4. $b^{\frac{5}{2}}$.	7. $d^{\frac{3}{5}}$.	10. $a^{\frac{m}{3}}$.
2. $a^{\frac{4}{3}}$.	5. $c^{\frac{8}{5}}$.	8. $m^{\frac{5}{9}}$.	11. $a^{\frac{1}{m}}$.
3. $a^{\frac{5}{6}}$.	6. $x^{\frac{7}{9}}$.	9. $n^{\frac{7}{8}}$.	12. $x^{\frac{m}{3}}$.

13. Write the third root of the fourth power of a.
14. Write the fifth power of the fourth root of x.
15. Write the eighth root of the twelfth power of y.

287. A ***Perfect Power*** is one whose exact root can be found. This root is called a *rational quantity*.

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285. How are roots denoted? 286. What does the numerator of a fractional exponent denote? The denominator? 287. What is a perfect power?

* From the Latin *radix*, a root.

The sign $\sqrt{}$ is a corruption of the letter r, the initial of *radix*.

288. An ***Imperfect Power*** is a quantity whose exact root cannot be found.

289. A ***Surd*** is the root of an imperfect power. It is often called an *irrational quantity.*

Thus, 5 is an imperfect power, and its square root, 2.23+, is a surd.

NOTE.—All *roots* as well as *powers* of 1, are 1. For, a root is a factor, which multiplied into itself produces a power; but no number except 1 multiplied into itself can produce 1.

Thus, 1, 1^3, 1^n, and $\sqrt{1}$, $\sqrt[3]{1}$, $\sqrt[n]{1}$, etc., are all equal.

290. *Negative Exponents* are used in expressing *roots* as well as *powers.* (Arts. 255, 257.)

Thus, $\frac{1}{a^{\frac{1}{2}}} = a^{-\frac{1}{2}}$; $\frac{1}{a^{\frac{1}{3}}} = a^{-\frac{1}{3}}$; $\frac{1}{a^{\frac{1}{n}}} = a^{-\frac{1}{n}}$.

291. The *value* of a quantity is not altered if the *index* of the power or root is exchanged for any other index of the *same value.*

Thus, instead of $x^{\frac{1}{2}}$, we may employ $x^{\frac{2}{4}}$, etc, Hence,

292. A *fractional* exponent may be expressed in *decimals.*

Thus, $a^{\frac{1}{2}} = a^{\frac{5}{10}} = a^{0.5}$. That is, the square root of a is equal to the fifth power of the tenth root of a.

16. Write $a^{\frac{1}{4}}$ in decimals.
17. Write $a^{\frac{1}{5}}$ in decimals.
18. Write $a^{\frac{3}{5}}$ in decimals.
19. Write $b^{\frac{4}{5}}$ in decimals.
20. Write $x^{\frac{7}{8}}$ in decimals.
21. Write $y^{\frac{3}{4}}$ in decimals.
22. Express $a^{\frac{1}{3}}$ in decimals. *Ans.* $a^{\frac{1}{3}} = a^{0.333333+}$.
23. Express $x^{\frac{2}{3}}$ in decimals. *Ans.* $x^{\frac{2}{3}} = x^{0.66666+}$.
24. Express $y^{\frac{5}{4}}$ in decimals. *Ans.* $y^{\frac{5}{4}} = y^{1.25}$.
25. Write $a^{\frac{8}{5}}$ in decimals. *Ans.* $a^{\frac{8}{5}} = a^{1.6}$.

NOTES.—1. In many cases, a *fractional exponent* can only be expressed *approximately* by *decimals.*

2. *Decimal indices* form the basis of *logarithms.*

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288. An imperfect power? 289. A surd? 290. Are negative exponents used in expressing roots? 292. How are fractional exponents sometimes expressed? *Note.* Of what do decimal indices form the basis?

293. The ***Signs of Roots*** are governed by the following

PRINCIPLES.

1°. ***An odd root of a quantity has the same sign as the quantity.***

2°. *An even root of a positive quantity is either positive or negative, and has the double sign,* $\pm$.

Thus, the square of $+a$ is a^2, and the square of $-a$ is a^2; therefore the square root of a^2 may be either $+a$ or $-a$; that is, $\sqrt{a^2} = \pm a$.

3°. *The root of the product of several factors is equal to the product of their roots.*

NOTES.—1. The *ambiguity* of an *even* root is removed, when it is known whether the *power* arises from a *positive* or a *negative* quantity.

2. It should also be observed that the two square roots of a *positive* quantity are *numerically equal*, but have *contrary* signs.

294. An ***Even Root*** of a *negative* quantity cannot be found. It is therefore said to be *impossible*.

Thus, the square root of $-a^2$ is neither $+a$ nor $-a$. For, $+a \times +a = +a^2$; and $-a \times -a = +a^2$. Hence,

295. *An even root of a negative quantity is called an Imaginary Quantity.*

Thus, $\sqrt{-4}$, $\sqrt{-a^2}$, $\sqrt[4]{-a^2}$, are imaginary quantities.

296. To Find the *Root* of a Monomial.

1. What is the square root of a^2?

ANALYSIS.—Since $a^2 = a \times a$, it follows that one of the equal factors of a^2 is a; therefore, a is its square root. (Art. 283.)

OPERATION.

$$\sqrt{a^2} = a$$

Again, since multiplying the index of a quantity by a number raises the quantity to a corresponding power, it follows that dividing the index by the same number resolves the quantity into a corresponding root. Thus, dividing the index of a^2 by 2, we have a^1 or a, which is the square root of a^2.

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293. What principles govern the signs of roots? When is the double sign used? Illustrate this. *Note.* When is the ambiguity removed? 294. What is an even root of a negative quantity? Illustrate. 295. What is it called?

2. What is the square root of $9a^4b^2$?

ANALYSIS.—Since $9 = 3 \times 3$, the index of $a^4 = 2 \times 2$, and the index of $b^2 = 1 \times 2$, it follows that the square root of 9 is 3, that of a^4 is a^2, and that of b^2 is b^1 or b. Therefore, $\sqrt{9a^4b^2} = 3a^2b$. Hence, the

OPERATION.

$$\sqrt{9a^4b^2} = 3a^2b$$

RULE.—*Divide the index of each letter by the index of the required root; to the result prefix the root of the coefficient with the proper sign.* (Art. 293.)

NOTES.—1. The sign $-$ must be prefixed to the odd root of every negative quantity; the double sign $\pm$ is usually prefixed to even roots. (Art. 293, Note.)

2. This rule is based upon Principle 3. If a quantity is an imperfect power, its root can only be indicated.

Find the required roots of the following quantities:

3.	$\sqrt{a^6}$.	10.	$\sqrt{36a^4b^2}$.
4.	$\sqrt[4]{a^1 \text{ or } a}$.	11.	$\sqrt[3]{2x^2y^2}$.
5.	$\sqrt[3]{4xy}$.	12.	$\sqrt{64a^8b^6}$.
6.	$\sqrt[3]{8a^5b^6}$.	13.	$\sqrt[5]{13xy}$.
7.	$\sqrt[3]{27abc}$.	14.	$\sqrt{49x^4y^6}$.
8.	$\sqrt[4]{16a^{4m}}$.	15.	$\sqrt[3]{27a^2b^3}$.
9.	$\sqrt[5]{3a^8x^{10}}$.	16.	$\sqrt{\dfrac{49x^4}{64y^2}}$.

297. To Extract the *Square Root* of the Square of a Binomial.

1. Required the square root of $a^2 + 2ab + b^2$.

ANALYSIS. — Arrange the terms according to the powers of the letter a; the square root of the first term is a, which is the first term of the root. Next, subtracting its square from the given quantity, bring down the remainder, $2ab + b^2$.

OPERATION.

$$\begin{array}{r|l} & a^2 + 2ab + b^2\ (\,a + b \\ & a^2 \\ \hline 2a + b\,) & 2ab + b^2 \\ & 2ab + b^2 \\ \hline \end{array}$$

296. How find the root of a monomial? *Notes.*—1. What is the root of a fraction? 2. Upon what principle is this rule based?

Divide this remainder by $2a$, or double the root thus found, and the quotient b is the other term of the root. Place b both in the root and on the right of the divisor. Finally, multiply the divisor, thus increased, by the second term of the root, and subtracting the product from $2ab+b^2$, there is no remainder. Therefore, $a+b$ is the root required.

The square root of $a^2-2ab+b^2$ is found in the same manner, the terms of the root being connected by the sign $-$. Hence, the

RULE.—*Find the square roots of the first and third terms, and connect them by the sign of the middle term.*

2. What is the square root of $x^2 + 4x + 4$?
3. What is the square root of $a^2 - 2a + 1$?
4. What is the square root of $1 + 2x + x^2$?
5. What is the square root of $x^2 + \frac{4}{3}x + \frac{4}{9}$?
6. What is the square root of $a^2 - a + \frac{1}{4}$?
7. What is the square root of $x^2 + ab + \frac{b^2}{4}$?

298. To Extract the Square Root of a Polynomial.

8. Required the square root of $4a^4-12a^3+5a^2+6a+1$.

OPERATION.

$$\begin{array}{rl}
 & 4a^4 - 12a^3 + 5a^2 + 6a + 1 \;(\; 2a^2 - 3a - 1 \\
 & \underline{4a^4 \qquad\qquad\qquad\qquad\qquad} \\
4a^2 - 3a \;) & - 12a^3 + 5a^2 + 6a + 1 \\
 & \underline{- 12a^3 + 9a^2 \qquad\qquad} \\
4a^2 - 6a - 1 \;) & - 4a^2 + 6a + 1 \\
 & \underline{- 4a^2 + 6a + 1}
\end{array}$$

ANALYSIS.—The square root of the first term is $2a^2$, which is the first term of the root. Subtract its square from the whole quantity and bring down the remaining terms. Divide the remainder by double the root thus found; the quotient $-3a$ is the next term of the root, and is placed both in the root and on the right of the partial divisor. Multiply the divisor thus increased by the term last placed in the root, and subtract the product as before.

Next, divide the remainder by twice the part of the root already found, and the quotient is -1, which is placed both in the root and on the right of the divisor.

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297. How extract the square root of the square of a binomial?

Finally, multiply the divisor, thus increased, by the term last placed in the root, and subtracting the product, as before, there is no remainder. Therefore, the required root is $2a^2-3a-1$. Hence, the

RULE.—I. *Arrange the terms according to the powers of some letter, find the square root of the first term for the first term of the root, and subtract its square from the given quantity.*

II. *Divide the remainder by double the root already found, and place the quotient both in the root and on the right of the divisor.*

III. *Multiply the divisor thus increased by the term last placed in the root, and subtract the product from the last dividend. If there is a remainder, proceed with it as before, till the root of all the quantities is found.*

PROOF.—*Multiply the root by itself, as in arithmetic.*

NOTE.—This rule is essentially the same as that used for extracting the *square root* of numbers.

Extract the square root of the following quantities:

9. $x^2 + 2xy + y^2 + 2xz + 2yz + z^2$.
10. $a^2 - 4ab + 2a + 4b^2 - 4b + 1$.
11. $a^4 + 4a^2b + 4b^2 - 4a^2 - 8b + 4$.
12. $1 - 4b^2 + 4b^4 + 2x - 4b^2x + x^2$.
13. $4a^4 - 16a^3 + 24a^2 - 16a + 4$.
14. $a^2 - ab + \frac{1}{4}b^2$.
15. $\dfrac{x^2}{y^2} - 2 + \dfrac{y^2}{x^2}$.

299. The *fourth* root of a quantity may be found by extracting the square root *twice;* that is, by extracting the square root of the square root.

Thus, $\sqrt{16a^4} = 4a^2$, and $\sqrt{4a^2} = 2a$. Therefore, $2a$ is the fourth root of $16a^4$.

PROOF. $2a \times 2a = 4a^2$; $4a^2 \times 4a^2 = 16a^4$.

The *eighth,* the *sixteenth,* etc., roots may be found in like manner.

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298. Of a polynomial? 299. How find the fourth root of a quantity? The eighth?

CHAPTER XIV.

RADICAL QUANTITIES.

300. A ***Radical*** is the root of a quantity indicated by the radical sign or fractional exponent.

NOTES.—1. The *figures* or *letters* placed before radicals are *coefficients*.

2. In the following investigations, all quantities placed under the radical sign, or having a fractional exponent, whether *perfect* or *imperfect* powers, are treated as *radicals*, unless otherwise mentioned.

301. The ***Degree*** of a radical is denoted by its *index*, or by the *denominator* of its fractional exponent. (Arts. 285, 286.)

Thus, $\sqrt[3]{ax}$, $a^{\frac{1}{3}}$, and $(a+b)^{\frac{1}{3}}$, are radicals of the same degree.

302. ***Like Radicals*** are those which express the *same* root of the *same* quantity. Hence, *like* radicals are *like* quantities. (Art. 43.)

Thus, $5\sqrt{a^2-b}$ and $3\sqrt{a^2-b}$, etc., are like radicals.

REDUCTION OF RADICALS.

303. ***Reduction of Radicals*** is changing their form without altering their value.

304. The ***Simplest Form*** of radicals is that which contains no *factor* whose indicated root can be extracted. Hence, in reducing them to their simplest form, *all exact powers* of the same name as the root must be removed from under the radical sign.

---

300. What is a radical? 301. How is the degree of a radical denoted? 302. What are like radicals? 303. Define reduction of radicals. 304. What is the simplest form of radicals?

CASE I.

305. To Reduce a Radical to its Simplest Form.

1. Reduce $\sqrt{18a^2x}$ to its simplest form.

OPERATION.

$$\sqrt{18a^2x} = \sqrt{9a^2 \times 2x}$$
$$" \quad = \sqrt{9a^2} \times \sqrt{2x}$$
$$\therefore \sqrt{18a^2x} = 3a\sqrt{2x}$$

ANALYSIS.—By inspection, we perceive that the given radical is composed of two factors, $9a^2$ and $2x$, the first being a perfect square and the second a surd. (Art. 289.) Removing $9a^2$ from under the radical sign and extracting its square root, we have $3a$, which prefixed to the other factor gives $3a\sqrt{2x}$, the simplest form required.

2. Reduce $4\sqrt[3]{a^4 - a^3x}$ to its simplest form.

OPERATION.

$$4\sqrt[3]{a^4-a^3x} = 4\sqrt[3]{a^3 \times (a-x)}$$
$$" \quad = 4\sqrt[3]{a^3} \times \sqrt[3]{a-x}$$
$$\therefore 4\sqrt[3]{a^4-a^3x} = 4a\sqrt[3]{a-x}$$

ANALYSIS. — Factoring the radical part, we have the two factors, $\sqrt[3]{a^3}$ and $\sqrt[3]{a-x}$, the first being a perfect cube, and the second a surd. Remove a^3 from under the radical sign, and its root is a, which multiplied by the coefficient 4, and prefixed to the radical part, gives $4a\sqrt[3]{a-x}$, the simplest form. Hence, the

RULE.—I. *Resolve the radical into two factors, one of which is the greatest power of the same name as the root.*

II. *Extract the root of this power, and multiplying it by the coefficient, prefix the result to the other factor, with the radical sign between them.*

NOTES.—1. This rule is based upon the principle that the root of the product of two or more factors is equal to the product of their roots.

2. When the radicals are small, the greatest exact power they contain may be readily found by inspection.

3. Reduce $3\sqrt{50a^2x}$ to its simplest form. *Ans.* $15a\sqrt{2x}$.

4. Reduce $6\sqrt[3]{54x^3y}$ to its simplest form. *Ans.* $18x\sqrt[3]{2y}$.

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305. Recite the rule. *Note.* Upon what based?

306. To Find in large Radicals the Greatest Power corresponding to the indicated Root.

5. Reduce $\sqrt{1872}$ to its simplest form.

OPERATION.

$$\begin{array}{l} 4\,)\,\underline{1872} \\ 4\,)\,\underline{468} \\ 9\,)\,\underline{117}\,(\,13 \end{array} \qquad \begin{aligned} \sqrt{1872} &= \sqrt{4\times4\times9}\times\sqrt{13} \\ \text{"} \quad &= \sqrt{144}\times\sqrt{13} \\ \sqrt{1872} &= 12\sqrt{13},\ \textit{Ans.} \end{aligned}$$

ANALYSIS.—Divide the radical by the smallest power of the same degree that is a factor of it; the quotient is 468. Divide this quantity by 4; the second quotient is 117. The smallest power of the same degree that will divide 117, is 9. The quotient is 13, which is not divisible by any power of the same degree. The product of the divisors, $4\times4\times9 = 144$, is the greatest square of the given radical. Extracting the square root of 144, we have $12\sqrt{13}$, the simplest form required. Hence, the

RULE.—*Divide the radical by the smallest power of the same degree which is a factor of the given radical.*

Divide this quotient as before; and thus proceed till a quotient is obtained which is not divisible by any power of the same degree. The continued product of the divisors will be the greatest power required.

NOTE.—This rule is founded on the principle that the *product* of any two or more *square* numbers is a *square*, the *product* of any two or more *cubic* numbers is a *cube*, etc.

Thus, $2^2\times3^2 = 36 = 6^2$; and $2^3\times3^3 = 216 = 6^3$.

Reduce the following radicals to their simplest form:

6.	$\sqrt{a^2b}$.	12.	$\sqrt[3]{54a^3c}$.
7.	$\sqrt{8a^2b}$.	13.	$7\sqrt{9a^2-27a^2b}$.
8.	$2\sqrt{9xy}$.	14.	$\sqrt[3]{64x^3y}$.
9.	$3\sqrt[3]{24}$.	15.	$\sqrt[4]{81a^4b}$.
10.	$5\sqrt[3]{135}$.	16.	$\sqrt{468a^2c}$.
11.	$6\sqrt{252a^2b}$.	17.	$\sqrt{1584a^2}$.

306. How find the greatest power corresponding to the indicated root, in large radicals? *Note.* On what principle is this rule founded?

CASE II.

307. To Reduce a Rational Quantity to the Form of a *Radical.*

1. Reduce $3a^2$ to the form of the cube root.

OPERATION.

$$(3a^2)^3 = 27a^6$$
$$\therefore\ 3a^2 = \sqrt[3]{27a^6}$$

ANALYSIS.—The cube root of a quantity, we have seen, is one of its three equal factors. (Art. 284.) Now $3a^2$ raised to the third power is $27a^6$. Therefore $3a^2 = \sqrt[3]{27a^6}$. Hence, the

RULE.—*Raise the quantity to the power denoted by the given root, and to the result prefix the corresponding radical sign.*

NOTE.—The coefficient of a radical, or any factor of it, may be placed under the sign, by raising it to the *corresponding power*, and placing it as a factor under the radical sign.

2. Reduce $2a^3b$ to the form of the cube root.
3. Reduce $(2a + b)$ to the form of the square root.
4. Reduce $(a - 2b)$ to the form of the square root.
5. Place the coefficient of $3a\sqrt{b}$ under the radical sign.
6. Place the coefficient of $2a^2\sqrt[3]{ab}$ under the radical sign.
7. Reduce $2x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}}$ to the form of the fourth root.
8. Reduce $\frac{1}{3}abc$ to the form of the cube root.
9. Reduce $3(a - b)$ to the form of the cube root.
10. Reduce a^2 to the form of the cube root.

NOTE.—When a power is to be raised to the form of a required root, it is not the given *letter* that is to be raised, but the *power* of the letter.

11. Reduce a^3c^2 to the form of the fourth root.
12. Reduce $a - b$ to the form of the square root.
13. Reduce a^m to the form of the nth root.

---

307. How reduce a rational quantity to the form of a radical? *Note.* How place a coefficient under the radical sign? *Note.* How raise a power to the form of a required root?

CASE III.

308. To Reduce Radicals of different Degrees to others of equal Value, having a *Common Index*.

1. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{4}}$ to equivalent radicals of a common index.

ANALYSIS.—The fractional indices $\frac{1}{3}$ and $\frac{1}{4}$, reduced to a common denominator become $\frac{4}{12}$ and $\frac{3}{12}$. But $a^{\frac{4}{12}} = (a^4)^{\frac{1}{12}}$, and $b^{\frac{3}{12}} = (b^3)^{\frac{1}{12}}$. (Art. 174.)

OPERATION.

$$\tfrac{1}{3} = \tfrac{4}{12} \text{ and } \tfrac{1}{4} = \tfrac{3}{12}$$
$$\therefore a^{\frac{1}{3}} = a^{\frac{4}{12}} \text{ and } b^{\frac{1}{4}} = b^{\frac{3}{12}}$$
$$a^{\frac{4}{12}} = (a^4)^{\frac{1}{12}} \text{ and } b^{\frac{3}{12}} = (b^3)^{\frac{1}{12}}$$

Therefore $(a^4)^{\frac{1}{12}}$ and $(b^3)^{\frac{1}{12}}$ are the radicals required. Hence, the

RULE.—I. *Reduce the indices to a common denominator.*
II. *Raise each quantity to the power expressed by the numerator of the new index, and indicate the root expressed by the common denominator.* (Art. 174.)

Reduce the following radicals to a common index:

2. $a^{\frac{1}{3}}$ and $(bc)^{\frac{2}{5}}$.
3. $3^{\frac{1}{2}}$ and $5^{\frac{3}{4}}$.
4. $a^{\frac{1}{2}}$ and $6^{\frac{2}{3}}$.
5. $\sqrt{5}$, $\sqrt[3]{3}$, and $\sqrt[4]{2}$.
6. $\sqrt[3]{2x^4}$ and $\sqrt{5x^3}$.
7. $\sqrt{4a^3}$ and $\sqrt[3]{2a^2}$.
8. $a^{\frac{1}{m}}$ and $b^{\frac{1}{n}}$.
9. $b^{\frac{1}{3}}$ and $c^{\frac{1}{n}}$.
10. $(a+b)^{\frac{1}{2}}$ and $(a-b)^{\frac{1}{3}}$.
11. $(x-y)^{\frac{2}{3}}$ and $(x+y)^{\frac{3}{4}}$.

CASE IV.

309. To Reduce a Quantity to any *Required Index*.

1. Reduce $a^{\frac{1}{6}}$ to the index $\frac{1}{2}$.

ANALYSIS.—Divide the index $\frac{1}{6}$ by $\frac{1}{2}$; we have $\frac{2}{6}$ or $\frac{1}{3}$. Place this index over a; it becomes $a^{\frac{1}{3}}$, and setting the required index over this, the result, $(a^{\frac{1}{3}})^{\frac{1}{2}}$, is the answer. Hence, the

OPERATION.

$$\tfrac{1}{6} \div \tfrac{1}{2} = \tfrac{1}{6} \times \tfrac{2}{1} = \tfrac{2}{6}$$
$$\tfrac{1}{6} \div \tfrac{1}{2} = \tfrac{2}{6} = \tfrac{1}{3}$$
$$\therefore (a^{\frac{1}{3}})^{\frac{1}{2}},\ Ans.$$

---

308. How reduce radicals to a common index?

RULE.—*Divide the index of the given quantity by the required index, and placing the quotient over the quantity, set the required index over the whole.*

NOTE.—This operation is the same as resolving the original index into *two factors*, one of which is the required index. (Art. 126, *note*.)

2. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{2}{7}}$ to the index $\frac{1}{3}$.

SOLUTION. $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2}$, the first index.
$\frac{2}{7} \div \frac{1}{3} = \frac{2}{7} \times \frac{3}{1} = \frac{6}{7}$, the second index.

Therefore, $(a^{\frac{3}{2}})^{\frac{1}{3}}$ and $(b^{\frac{6}{7}})^{\frac{1}{3}}$ are the quantities required.

3. Reduce $3^{\frac{1}{4}}$ and $4^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.
4. Reduce a^2 and b^3 to the common index $\frac{1}{3}$.
5. Reduce a^3 and $b^{\frac{1}{3}}$ to the common index $\frac{3}{4}$.
6. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{2}{3}}$ to the common index $\frac{5}{6}$.
7. Reduce $a^{\frac{1}{m}}$ and $b^{\frac{2}{m}}$ to the common index $\frac{1}{n}$.

ADDITION OF RADICALS.

310. To Find the *Sum* of two or more Radicals.

1. What is the sum of $3\sqrt{a}$ and $5\sqrt{a}$?

ANALYSIS.—Since these radicals are of the same degree and have the same radical part, they are like quantities. (Art. 43.) Therefore their coefficients may be added in the same manner as rational quantities. (Art. 67.)

OPERATION.

$$3\sqrt{a} + 5\sqrt{a} = 8\sqrt{a}$$

2. What is the sum of $3\sqrt{8}$ and $4\sqrt{18}$?

ANALYSIS.—These radicals are of the *same degree*, but the radical parts are *unlike*; therefore, they cannot be united in their present form. Reducing them to their simplest form, we have $3\sqrt{8} = 6\sqrt{2}$, and $4\sqrt{18} = 12\sqrt{2}$, which are like radicals. (Arts. 302, 305.) Now $6\sqrt{2}$ and $12\sqrt{2} = 18\sqrt{2}$, *Ans.*

OPERATION.

$$3\sqrt{8} = 3\sqrt{4} \times \sqrt{2} = 6\sqrt{2}$$
$$4\sqrt{18} = 4\sqrt{9} \times \sqrt{2} = 12\sqrt{2}$$
$$\therefore 6\sqrt{2} + 12\sqrt{2} = 18\sqrt{2}$$

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309. How reduce quantities to any required index? *Note.* To what is this operation similar?

3. What is the sum of $3\sqrt{18}$ and $4\sqrt[3]{24}$.

ANALYSIS. — Reducing the radicals to their simplest form, we have $3\sqrt{18} = 9\sqrt{2}$, and $4\sqrt[3]{24} = 8\sqrt[3]{3}$, which are unlike quantities, and can only be added by writing them one after the other, with their proper signs. (Arts. 43, 67.) Hence, the

OPERATION.

$$3\sqrt{18} = 3\sqrt{9} \times \sqrt{2} = 9\sqrt{2}$$
$$4\sqrt[3]{24} = 4\sqrt[3]{8} \times \sqrt[3]{3} = 8\sqrt[3]{3}$$
$$Ans.\ 9\sqrt{2} + 8\sqrt[3]{3}$$

RULE.—I. *Reduce the radicals to their simplest form.*

II. *If the radical parts are alike, add the coefficients, and to the sum annex the common radical.*

If the radicals are unlike, write them one after another, with their proper signs.

NOTE.—To determine whether radicals are *alike*, it is generally necessary to reduce them to their simplest form. (Art. 305.)

Find the sum of the following radicals:

4. $\sqrt{12}$ and $\sqrt{27}$.
5. $\sqrt{20}$ and $\sqrt{48}$.
6. $2\sqrt{b^3}$ and $3\sqrt{a^2b}$.
7. $a\sqrt{3a^3b}$ and $c\sqrt{27ab}$.
8. $3\sqrt{18ax^2}$ and $2\sqrt{32a^3}$.
9. $3\sqrt[3]{54}$ and $4\sqrt[3]{128}$.
10. $7\sqrt{243}$ and $5\sqrt{363}$.
11. $a\sqrt{81b}$ and $3a\sqrt{49b}$.
12. $b\sqrt{25x^2c}$ and $\sqrt{36x^4c}$.
13. $4\sqrt[3]{x^6y}$ and $5\sqrt{x^8y}$.

SUBTRACTION OF RADICALS.

311. To Find the *Difference* between two Radicals.

1. From $3\sqrt{45}$ subtract $2\sqrt{20}$.

ANALYSIS.—Reducing to the simplest form, we have $9\sqrt{5}$ and $4\sqrt{5}$, which are like quantities. Now $9\sqrt{5} - 4\sqrt{5} = 5\sqrt{5}$, the difference required. Hence, the

OPERATION.

$$3\sqrt{45} = 3\sqrt{9} \times \sqrt{5} = 9\sqrt{5}$$
$$2\sqrt{20} = 2\sqrt{4} \times \sqrt{5} = 4\sqrt{5}$$
$$\therefore 3\sqrt{45} - 2\sqrt{20} = 5\sqrt{5}$$

310. How add radicals? *Note.* How determine whether they are like quantities?

Rule.—*Reduce the radicals to their simplest form; change the sign of the subtrahend, and proceed as in addition of radicals.* (Art. 310.)

	(2.)	(3.)	(4.)
From	$4\sqrt{112}$	$\sqrt{480}$	$4\sqrt{320}$
Take	$\sqrt{448}$	$4\sqrt{63}$	$-5\sqrt{80}$

5. From $3\sqrt{49ax^3}$ take $2\sqrt{25ax}$.
6. From $5\sqrt[3]{a+b}$ take $3\sqrt[3]{a+b}$.
7. From $3\sqrt[3]{b}$ take $-4\sqrt[3]{b}$.
8. From $3\sqrt[3]{250b^4x}$ take $2\sqrt[3]{54b^4x}$.
9. From $-a^{-\frac{1}{2}}$ take $-2a^{-\frac{1}{2}}$.
10. From $5\sqrt{\frac{3}{4}}$ take $2\sqrt{\frac{1}{3}}$.

MULTIPLICATION OF RADICALS.

312. To *Multiply* Radical Quantities.

1. What is the product of $3\sqrt{a}$ by $2\sqrt{b}$.

Operation.

$$3\sqrt{a} \times 2\sqrt{b} = 6\sqrt{ab}$$

Analysis.—Since these radicals are of the same degree, we multiply the radical parts together, like rational quantities, and to the result prefix the product of the coefficients.

2. Multiply $3\sqrt{a}$ by $2\sqrt[3]{c}$.

Operation.

$$3\sqrt{a} = 3\left(a^{\frac{3}{6}}\right)$$
$$2\sqrt[3]{c} = 2\left(c^{\frac{2}{6}}\right)$$
$$Ans.\ 6\,(a^3c^2)^{\frac{1}{6}}$$

Analysis.—As these radicals are of different degrees, they cannot be multiplied together in their present form. We therefore reduce them to a common index, and then, multiplying as before, we have $6\sqrt[6]{a^3c^2}$.

3. Multiply $a^{\frac{1}{3}}$ by $a^{\frac{1}{2}}$.

Analysis.—These radicals are of different degrees, but of the same radical part or root; we therefore multiply them by adding their fractional exponents. $\frac{1}{3}+\frac{1}{2}=\frac{5}{6}$. Therefore, $a^{\frac{1}{3}} \times a^{\frac{1}{2}} = a^{\frac{5}{6}}$. Hence, the

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311. How subtract radicals?

RULE.—I. *Reduce the radicals to a common index.*

II.—*Multiply the radical parts together as rational quantities, and placing the result under the common index, prefix to it the product of the coefficients.*

NOTES.—1. *Roots of like quantities* are multiplied together by *adding their fractional exponents.* (Art. 94.)

2. This rule is based upon the principle that the *product* of the roots of two or more quantities is the *same* as the *root* of the *product.* (Art. 293, Prin. 3.)

3. The *product* of radicals becomes *rational,* whenever the numerator of the index can be divided by its denominator without a remainder.

4. If *rational* quantities are connected with *radicals* by the signs + or −, *each term* in the multiplicand must be multiplied by *each term* in the multiplier. (Art. 98.)

Multiply the following radicals:

4. $5\sqrt{18}$ by $3\sqrt{20}$.
5. $a\sqrt{x}$ by $b\sqrt{x}$.
6. $\sqrt{a+b}$ by $\sqrt{a-b}$.
7. $\sqrt{ax}$ by $\sqrt{cy}$.
8. $a^{\frac{2}{3}}$ by $c^{\frac{1}{4}}$.
10. $a^{\frac{1}{m}}$ by $x^{\frac{1}{n}}$.
11. $7\sqrt[3]{4}$ by $3\sqrt[3]{4}$.
12. $\sqrt{9a}$ by $\sqrt{16a}$.
13. $\sqrt{18}$ by $\sqrt{2}$.
14. $\sqrt{8ax}$ by $\sqrt{2ax}$.

(9.)

$$
\begin{array}{ll}
\text{Multiply} & a+\sqrt{b} \\
\text{By} & c+\sqrt{d} \\
\hline
 & ac+c\sqrt{b} \\
 & \quad +a\sqrt{d}+\sqrt{bd} \\
\hline
\textit{Ans.} & ac+c\sqrt{b}+a\sqrt{d}+\sqrt{bd}
\end{array}
$$

(15.)

$$
\begin{array}{ll}
\text{Mult.} & a+\sqrt{x} \\
\text{By} & 1+b\sqrt{x} \\
\hline
 & a+\sqrt{x} \\
 & \quad +ab\sqrt{x}+bx \\
\hline
\textit{Ans.} & a+\sqrt{x}+ab\sqrt{x}+bx
\end{array}
$$

16. $2\sqrt{\frac{5}{8}}$ by $2\sqrt{\frac{1}{2}}$.
17. $4\sqrt{\frac{5}{9}}$ by $3\sqrt{\frac{1}{4}}$.
18. $\sqrt[\frac{3}{2}]{m+n}$ by $\sqrt[\frac{2}{3}]{m+n}$.
19. $\sqrt{\dfrac{9ad}{2b}}$ by $\sqrt{\dfrac{2ab}{3c}}$.

---

312. How multiply radicals? *Notes.* How are roots of like quantities multiplied? Upon what principle is this rule based? When does the product of radicals become rational? If radicals are connected with rational quantities, how multiply them?

DIVISION OF RADICALS.

313. To *Divide* Radical Quantities.

1. Divide $4\sqrt{24ac}$ by $2\sqrt{8a}$.

OPERATION.

$$\frac{4\sqrt{24ac}}{2\sqrt{8a}} = 2\sqrt{3c}$$

ANALYSIS.—Since the given radicals are of the same degree, one may be divided by the other, like rational quantities, the quotient being $\sqrt{3c}$. (Art. 111.) To this result prefixing the quotient of one coefficient divided by the other, we have $2\sqrt{3c}$, the quotient required.

2. Divide $4\sqrt{ac}$ by $2\sqrt[3]{a}$.

$$\frac{4\sqrt{ac}}{2\sqrt[3]{a}} = \frac{4\,(ac)^{\frac{1}{2}}}{2\,(a)^{\frac{1}{3}}}$$

$$\therefore \frac{4\sqrt{ac}}{2\sqrt[3]{a}} = \frac{4\,(a^3c^3)^{\frac{1}{6}}}{2\,(a^2)^{\frac{1}{6}}}$$

$$\text{or } 2\,(ac^3)^{\frac{1}{6}}, \textit{ Ans.}$$

ANALYSIS.—Since these radicals are of different degrees, they cannot be divided in their present form. We therefore reduce them to a common index, then divide one by the other, and to the result prefix the quotient of the coefficients. The answer is $2\sqrt[6]{ac^3}$.

3. Divide $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$.

OPERATION.

$$a^{\frac{1}{2}} = a^{\frac{3}{6}}$$

$$a^{\frac{1}{3}} = a^{\frac{2}{6}}$$

$$a^{\frac{3}{6}} \div a^{\frac{2}{6}} = a^{\frac{1}{6}}$$

$$\therefore a^{\frac{1}{2}} \div a^{\frac{1}{3}} = a^{\frac{1}{6}}$$

ANALYSIS.—These radicals are of different degrees, but have the same radical part or root. We therefore divide them by subtracting the fractional exponent of the divisor from that of the dividend. (Art. 113.) Reducing the exponents to a common denominator, $a^{\frac{1}{2}} = a^{\frac{3}{6}}$, and $a^{\frac{1}{3}} = a^{\frac{2}{6}}$. Now $a^{\frac{3}{6}} \div a^{\frac{2}{6}} = a^{\frac{1}{6}}$, *Ans.* Hence, the

RULE.—I. *Reduce the radical parts to a common index.*

II. *Divide one radical part by the other, and placing the quotient under the common index, prefix to the result the quotient of their coefficients.*

NOTE.—*Roots of like quantities* are divided *by subtracting the fractional exponent of the divisor from that of the dividend.* (Art. 113.)

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313. How divide radicals? *Note.* How divide roots of like quantities?

Divide the following radicals:

4. $\sqrt{12a^3c}$ by $\sqrt{4c}$.	10. $14a\sqrt{xy}$ by $7\sqrt{y}$.
5. $6\sqrt{bdx^2}$ by $2\sqrt{dx}$.	11. $(a+b)^{\frac{2}{n}}$ by $(a+b)^{\frac{1}{n}}$.
6. $(a^3+ax)^{\frac{1}{3}}$ by $a^{\frac{1}{3}}$.	12. $3\sqrt{50x^4}$ by $\sqrt{2x}$.
7. $12(a^2y^2)^{\frac{1}{4}}$ by $(ay)^{\frac{1}{4}}$.	13. $\sqrt[3]{x^2-y^2}$ by $\sqrt[3]{x+y}$.
8. $24b\sqrt{ax}$ by $8\sqrt{a}$.	14. $16\sqrt{32}$ by $2\sqrt{4}$.
9. $15ac\sqrt{bx}$ by $2c\sqrt{x}$.	15. $8\sqrt{512}$ by $4\sqrt{2}$.

INVOLUTION OF RADICALS.

314. To Involve a Radical to any required Power.

1. Find the square of $a^{\frac{1}{3}}$.

OPERATION.

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{2}{3}}$$

$$\therefore a^{\frac{2}{3}}, \textit{ Ans.}$$

ANALYSIS.—As a square is the product of two equal factors, we multiply the given index by the index of the required power. Hence, the

RULE.—*Multiply the index of the root by the index of the required power, and to the result prefix the required power of the coefficient.*

NOTE.—A *root* is raised to a power of the *same name* by removing the radical sign or fractional exponent. (Ex. 2.)

2. Find the cube of $\sqrt[3]{a+b}$. *Ans.* $a+b$.
3. Find the cube of $a^{\frac{1}{4}}$.
4. What is the square of $3\sqrt{2x}$.
5. What is the cube of $2\sqrt[3]{a}$.
6. Required the cube of $\frac{x}{2}\sqrt{2x}$.
7. Required the cube of $4\sqrt{\frac{ax^2}{4}}$.
8. Find the fourth power of $3\sqrt{\frac{b}{3}}$.
9. What is the square of $a+\sqrt{y}$?

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314. How involve radicals to any required power? *Note.* How raise a root to a power of the same name?

EVOLUTION OF RADICALS.

315. To Extract the *Root* of a Radical.

1. Find the cube root of $a^3\sqrt[4]{b^2}$.

ANALYSIS.—Finding the root of a radical is the same in principle as finding the root of a rational quantity. (Art. 296.) Reducing the index of the radical to an equivalent fractional exponent, we extract the cube root by dividing it by 3. The result is $ab^{\frac{1}{6}}$. Hence, the

OPERATION.

$$\sqrt[3]{a^3\sqrt[4]{b^2}} = \sqrt[3]{a^3b^{\frac{2}{4}}}$$

$$\sqrt[3]{a^3b^{\frac{2}{4}}} = ab^{\frac{1}{6}}, \textit{ Ans.}$$

RULE.—*Divide the fractional exponent of the radical by the number denoting the required root, and to the result prefix the root of the coefficient.*

NOTES.—1. Multiplying the *index* of a radical by any number is the same as dividing the *fractional exponent* by that number.

Thus, $\sqrt[3]{a} = a^{\frac{1}{3}}$. Multiplying the former by 2, and dividing the latter by 2, we have $\sqrt[6]{a} = a^{\frac{1}{6}}$.

2. If the coefficient is not a perfect power, it should be placed under the radical sign and be reduced to its simplest form. (Art. 305.)

2. Required the square root of $9\sqrt[3]{a^2}$.
3. Required the square root of $4\sqrt[3]{3x}$.
4. Find the cube root of $3\sqrt{xy}$.
5. Find the cube root of $2b\sqrt{2b}$.
6. What is the cube root of $a\,(bc)^{\frac{1}{2}}$?
7. What is the fourth root of $\frac{4}{9}\sqrt[3]{\frac{4}{9}}$?
8. What is the fourth root of $\sqrt{a^3}\,\sqrt[3]{c^2}$?
9. Find the seventh root of $128\sqrt{a}$.
10. Find the fourth root of $\sqrt{a}\,(b^{\frac{1}{6}})$.
11. Find the fifth root of $4a^2\sqrt{2a}$.
12. Find the nth root of $a\sqrt[6]{bc}$.

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315. How extract the root of a radical? *Notes.* To what is multiplying the index of a radical equivalent? If the coefficient is not a perfect power, what is done?

REDUCING A RADICAL TO A RATIONAL QUANTITY.

CASE I.

316. To Reduce a Radical Monomial to a Rational Quantity.

1. Reduce $\sqrt{a}$ to a rational quantity.

ANALYSIS.—Since multiplying a root of a quantity into itself produces the quantity, it follows that $\sqrt{a} \times \sqrt{a} = a$, which is a rational quantity. (Art. 287.)

OPERATION.

$$\sqrt{a} \times \sqrt{a} = a$$

2. It is required to rationalize $a^{\frac{1}{3}}$.

ANALYSIS.—A root is multiplied by another root of the same quantity by adding the exponents; therefore we add to the index $\frac{1}{3}$ such a fraction as will make it equal to 1. (Art. 94.)

OPERATION.

$$a^{\frac{1}{3}} \times a^{\frac{2}{3}} = a$$

Thus, $a^{\frac{1}{3}} \times a^{\frac{2}{3}} = a^{\frac{1}{3}+\frac{2}{3}} = a^{\frac{3}{3}} = a$, the rational quantity required.

3. It is required to rationalize $x^{\frac{3}{4}}$.

SOLUTION.—Multiplying $x^{\frac{3}{4}}$ by $x^{\frac{1}{4}}$, the result is x, which is a rational quantity. Hence, the

OPERATION.

$$x^{\frac{3}{4}} \times x^{\frac{1}{4}} = x$$

RULE.—*Multiply the radical by the same quantity having such a fractional exponent as, when added to the given exponent, the sum shall be equal to a unit, or* 1.

4. Required a factor which will rationalize $a^{\frac{2}{5}}$.
5. What factor will rationalize $\sqrt[3]{a^2c}$?
6. What factor will rationalize $\sqrt[5]{(a+b)^2}$.
7. What factor will rationalize $\sqrt{a^3b^2c}$?
8. What factor will rationalize $\sqrt[3]{(x+y)^2}$?
9. What factor will rationalize $\sqrt[4]{(a+b)^3}$?
10. What factor will rationalize $\sqrt{(a+b+c)}$?

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316. How reduce a radical monomial to a rational quantity?

CASE II.

317. To Reduce a Radical Binomial to a Rational Quantity.

1. It is required to rationalize $\sqrt{a} + \sqrt{b}$.

ANALYSIS.—The product of the sum and difference of two quantities is equal to the difference of their squares (Art. 103); therefore, $(\sqrt{a}+\sqrt{b})$ multiplied by $(\sqrt{a}-\sqrt{b})$ $= a-b$, which is a rational quantity.

Therefore, the factor to employ as a multiplier is $\sqrt{a} - \sqrt{b}$.

OPERATION.

$$\begin{array}{r} \sqrt{a} + \sqrt{b} \\ \sqrt{a} - \sqrt{b} \\ \hline a + \sqrt{ab} \qquad \\ - \sqrt{ab} - b \\ \hline a - b, \textit{ Ans.} \end{array}$$

2. What factor will rationalize $\sqrt{x} - \sqrt{y}$?

ANALYSIS.—If the binomial $\sqrt{x} - \sqrt{y}$ is multiplied by the same terms with the sign of the latter changed to $+$, we have

$$(\sqrt{x} - \sqrt{y}) \times (\sqrt{x} + \sqrt{y}) = x - y.$$

(Art. 103.) Therefore, $\sqrt{x} + \sqrt{y}$ is the factor required. Hence, the

OPERATION.

$$\begin{array}{c} \sqrt{x} - \sqrt{y} \\ \sqrt{x} + \sqrt{y} \\ \hline x - y \end{array}$$

Ans. $\sqrt{x} + \sqrt{y}$

RULE.—*Multiply the binomial radical by the corresponding binomial with its connecting sign changed.*

3. What factor will rationalize $x + 4\sqrt{9}$?
4. Rationalize $\sqrt{9} - \sqrt{6}$.
5. What factor will rationalize $\sqrt{7} + \sqrt{a}$?
6. Rationalize $6 - \sqrt{5}$.
7. What factor will rationalize $\sqrt{3a} - \sqrt{3b}$.
8. Rationalize $\sqrt{a} - \sqrt{5}$.
9. What factor will rationalize $3\sqrt{a} + \sqrt{8}$.
10. What factor will rationalize $4\sqrt{2a} - 5\sqrt{b}$.

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317. How reduce a radical binomial to a rational quantity?

CASE III.

318. To Reduce a Radical Fraction to one whose Numerator or Denominator is a Rational Quantity.

1. Reduce $\frac{a}{\sqrt{b}}$ to a rational denominator.

ANALYSIS.—Multiply both terms of the fraction by the denominator $\sqrt{b}$, and the result is $\frac{a\sqrt{b}}{b}$, whose denominator is rational. (Art. 167, Prin. 3, *note*.) Hence, the

OPERATION.

$$\frac{a \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \frac{a\sqrt{b}}{b}$$

RULE.—*Multiply both terms of the fraction by such a factor as will make the required term rational.*

NOTE.—Since the product of the sum and difference of two quantities is equal to the difference of their squares, when the radical fraction is of the form $\frac{x}{\sqrt{a} - \sqrt{b}}$, if we multiply the terms by $(\sqrt{a} + \sqrt{b})$, we have $a - b$ for the denominator. (Art. 103.)

2. Rationalize the denominator of $\frac{3}{\sqrt{5}}$. *Ans.* $\frac{3\sqrt{5}}{5}$.

3. Rationalize the numerator of $\frac{\sqrt{a}}{\sqrt{x}}$. *Ans.* $\frac{a}{\sqrt{ax}}$.

4. Rationalize the denominator of $\frac{c}{\sqrt[3]{x}}$.

5. Rationalize the denominator of $\frac{a}{\sqrt[3]{c}}$.

6. Rationalize the denominator of $\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$.

7. Rationalize the denominator of $\frac{x}{\sqrt{a} - \sqrt{c}}$.

8. Rationalize the denominator of $\frac{1}{1 + \sqrt{3}}$.

9. Rationalize the denominator of $\frac{\sqrt{3}}{3 - \sqrt{3}}$.

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318. How reduce a radical fraction to one whose numerator or denominator is a rational quantity? When the fractions contain compound quantities, what principle enters into their reduction?

RADICAL EQUATIONS.

319. A ***Radical Equation*** is one in which the unknown quantity is under the radical sign.

320. To Solve a Radical Equation.

1. Given $\sqrt{x} + 2 = 7$, to find x.

ANALYSIS.—Transposing 2, we have, $\sqrt{x} = 5$. Since 5 is equal to the $\sqrt{x}$, it follows that the square of 5, or 25, must be the square of $\sqrt{x}$. Therefore, $x = 25$.

$$\sqrt{x} + 2 = 7$$
$$\sqrt{x} = 7 - 2 = 5$$
$$x = 5^2 = 25$$

2. Given $2a + \sqrt{x} = 9a$, to find x.

SOLUTION.—By the problem, $2a + \sqrt{x} = 9a$

By transposing, $\sqrt{x} = 7a$

By involution, $x = 49a^2$

3. Given $5\sqrt[3]{x+1} = 35$, to find x.

SOLUTION.—By the problem, $5\sqrt[3]{x+1} = 35$

Removing coefficient, $\sqrt[3]{x+1} = 7$

Involving, $x + 1 = 343$

Transposing, $x = 342$. Hence, the

RULE.—*Involve both sides to a power of the same name as the root denoted by the radical sign.*

NOTE.—Before *involving* the quantities, it is generally best to clear of fractions, and transpose the terms, so that the quantities under the radical sign shall stand alone on one side of the equation.

Reduce the following radical equations:

4. $a + \sqrt{x} + c = d$.	8. $\sqrt[3]{2x+3-6} = 13$.
5. $\sqrt[3]{x+2} = 3$.	9. $\sqrt[3]{x-4} = 3$.
6. $3\sqrt{x-4} + 5 = 7\frac{1}{2}$.	10. $2\sqrt[4]{x-5} = 4$.
7. $3\sqrt{\frac{x}{4}} = 24$.	11. $5\sqrt{\frac{x}{7}} = 30$.

319. What is a radical equation? 320. How solved? *Note.* What should be done before involving the quantities?

12. Given $\frac{a + \sqrt{2ax + b}}{b} = b^2$, to find x.

13. Reduce $\sqrt{a^2 + \sqrt{x}} = \frac{3 + c}{\sqrt{(a^2 + \sqrt{x})}}$.

ANALYSIS.—By removing the denominator the first member is squared. But x is still under the radical sign. This is removed by involving both members again.

OPERATION.

$$\sqrt{a^2 + \sqrt{x}} = \frac{3 + c}{\sqrt{(a^2 + \sqrt{x})}}$$

$$a^2 + \sqrt{x} = 3 + c$$

$$Ans.\ x = (3 + c - a^2)^2$$

14. Given $\frac{y - ay}{\sqrt{y}} = \frac{\sqrt{y}}{y}$, to find y.

15. Given $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{(a^2 + x^2)}}$, to find x.

NOTE.—If the equation has two radical expressions, connected with other terms by the signs + or −, it is advisable to transpose the terms so that one of the radicals shall stand alone on one side of the equation. By involving both members, one of the radicals becomes rational; and by repeating the operation, the other will also disappear.

16. Given $\sqrt{a + x} + \sqrt{a + x} = c$, to find x.

SOLUTION.—Given $\sqrt{a+x} + \sqrt{a+x} = c$

Transposing, $\sqrt{a+x} = c - \sqrt{a+x}$

Involving, $a+x = c^2 - 2c\sqrt{a+x} + a + x$

Transposing, $2c\sqrt{a+x} = c^2$

Dividing by $2c$, and involving, $a+x = \left(\frac{c}{2}\right)^2$

Transposing, $x = \left(\frac{c}{2}\right)^2 - a$, *Ans.*

17. Given $\sqrt{x + 12} = 2 + \sqrt{x}$, to find x.

18. Given $\sqrt{5} \times \sqrt{x + 2} = 2 + \sqrt{5x}$, to find x.

19. Given $\frac{\sqrt{x}}{x} = \frac{x - ax}{\sqrt{x}}$, to find x.

Note. If the equation has two radicals connected with other terms by + or −, what should be done?

CHAPTER XV.

QUADRATIC EQUATIONS.

321. Equations are divided into *different degrees,* as the first, second, third, etc., according to the powers of the unknown quantity contained in them.

An equation of the ***First Degree*** is called a *Simple Equation,* and contains only the *first* power of the unknown quantity.

An equation of the ***Second Degree*** is called a *Quadratic Equation,* and the highest power of the unknown quantity it contains is a *square.*

An equation of the ***Third Degree*** is called a *Cubic Equation,* and the highest power of the unknown quantity it contains is a *cube.*

An equation of the ***Fourth Degree*** is called a *Biquadratic,* etc.

322. ***Quadratic Equations*** are divided into *pure* and *affected.*

323. A ***Pure Quadratic*** contains the *square* only of the unknown quantity; as, $x^2 = b$.

324. An ***Affected Quadratic*** contains both the *first* and *second* powers of the unknown quantity; as, $x^2 + ax = cd$.

NOTES.—1. *Pure* quadratics are sometimes called *incomplete* equations; and *affected* quadratics, *complete* equations.

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321. How are equations divided? What is an equation of the first degree? The second? Third? Fourth? 322. How are quadratic equations divided? 325. What is a pure quadratic? 324. An affected quadratic? *Note.* What are they sometimes called?

2. Since the first member of a *pure* quadratic is always a complete square, and the first member of an *affected* quadratic is always an incomplete square, there seems to be an incongruity in calling the former an *incomplete* equation, and the latter a *complete* equation. The distinction is calculated to *confuse* rather than *enlighten* the pupil.

PURE QUADRATICS.

325. Every *pure quadratic* may be reduced to the form

$$x^2 = a.$$

For, by transposition, etc., all the terms containing x^2 can be reduced to one term, as bx^2; and all the known quantities to one term, as c. Then will

$$bx^2 = c.$$

Dividing both members by b, and substituting a for the quotient of $c \div b$, the result is the form,

$$x^2 = a.$$

326. Pure quadratic equations have *two roots,* which are the *same* numerically, but have *opposite* signs. (Art. 293, *n.*)

Thus, the square of $+a$ and of $-a$ is equally a^2. Hence,

$$\sqrt{a^2} = \pm a.$$

327. To Solve a *Pure Quadratic* Equation.

1. Find the value of x in $\frac{5x^2}{9} - 6 = \frac{x^2}{3} + 2.$

SOLUTION.—Given $\frac{5x^2}{9} - 6 = \frac{x^2}{3} + 2$

Clearing of fractions, $5x^2 - 54 = 3x^2 + 18$

Transposing, etc., $2x^2 = 72$

Removing coefficient, $x^2 = 36$

Extracting sq. root, $x = \pm 6$, *Ans.*

Substituting b for 2, and c for 72, in the third equation, we have the form, $bx^2 = c.$

Removing coefficient, etc., $x^2 = a.$ Hence, the

RULE.—*Reduce the given equation to the form $x^2 = a$, and extract the square root of both members.* (Art. 296.)

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326. How many roots has a pure quadratic?

Find the value of x in the following equations:

2. $3x^2 - 5 = 70$.
3. $9x^2 + 8 = 3x^2 + 62$.
4. $5x^2 + 9 = 2x^2 + 57$.
5. $6x^2 + 5 = 4x^2 + 55$.
6. $\dfrac{5x^2}{4} + 35 = 3x^2 + 7$.
7. $\dfrac{2x^2 + 8}{10} = \dfrac{x^2 - 6}{10} + 5$.
8. $\dfrac{x}{4} = \dfrac{x}{2} - \dfrac{4}{x}$.
9. $\dfrac{x}{2} + \dfrac{2}{x} = \dfrac{x}{3} + \dfrac{3}{x}$.
10. $2x^2 + 12 = 3x^2 - 37$.
11. $7x^2 - 7 = 3x^2 + 9$.
12. $a^2x^2 = a^4$.
13. $(x + 2)^2 = 4x + 5$.
14. $x^2 - 1 = \dfrac{6x^2 - 12}{4}$.
15. $\dfrac{x(2x + 9)}{30} = \dfrac{3x + 6}{10}$.
16. $\dfrac{5}{4 - x} + \dfrac{5}{4 + x} = \dfrac{8}{3}$.
17. $\dfrac{ax^2(a - 2)}{1 + x} = 1 - x$.

328. Radical equations, when cleared of radicals, often become pure quadratics.

18. Given $\sqrt{x^2 + 11} = \sqrt{2x^2 - 5}$, to find x.

SOLUTION.—Clearing of radicals, $x^2 + 11 = 2x^2 - 5$
Transposing and extracting root, $x = \pm 4$

19. Given $2\sqrt{x^2 - 5} = \dfrac{4x}{3}$, to find x.

20. Given $2\sqrt{x^2 - 4} = 4\sqrt{a^2 - 1}$, to find x.

21. Given $\sqrt{x + c} = \dfrac{d}{\sqrt{x - c}}$, to find x.

22. Given $\sqrt{\dfrac{5x^2 - 1}{x}} = \sqrt{x}$, to find x.

23. Given $\dfrac{b}{\sqrt{(x - a)}} = \sqrt{x + a}$, to find x.

24. Given $\dfrac{24}{\sqrt{x + 10}} = \sqrt{x - 10}$, to find x.

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327. What is the rule for the solution of pure quadratics? 328. What may radical equations become?

PROBLEMS

1. The product of one-third of a number multiplied by one-fourth of it is 108. What is the number?

2. What number is that, the fourth part of whose square being subtracted from 25, leaves 9?

3. How many rods on one side of a square field whose area is 10 acres?

4. A gentleman exchanges a rectangular piece of land 50 rods long and 18 wide, for one of equal area in a square form. Required the length of one side of the square.

5. Find two numbers that are to each other as 2 to 5, and whose product is 360.

6. If the number of dollars which a man has be squared and 7 be subtracted, the remainder is 29. How much money has he?

7. Find a number whose eighth part multiplied by its fifth part and the product divided by 16, will give a quotient of 10.

8. The product of two numbers is 900, and the quotient of the greater divided by the less is 4. What are the numbers?

9. A merchant buys a piece of silk for $40.50, and the price per yard is to the number of yards as 3 to 54. Required the number of yards and the price of each.

10. Find a number such that if 3 times the square be divided by 4 and the quotient be diminished by 12, the remainder will be 180.

11. A reservoir whose sides are vertical holds 266,112 gallons of water, is 6 feet deep, and square on the bottom. Required the length of one side, allowing 231 cubic inches to the gallon.

12. What number is that, to which if 10 be added, and from which if 10 be subtracted, the product of the sum and difference will be 156?

AFFECTED QUADRATICS.*

329. An ***Affected Quadratic Equation*** is one which contains the *first* and *second* powers of the unknown quantity; as, $ax^2 + bx = c$.

330. Every affected quadratic may be reduced to the form,

$$x^2 \pm ax = b,$$

in which a, b, and x may denote any quantity, either *positive* or *negative*, *integral* or *fractional*.

For, by transposition, etc., all the terms containing x^2 can be reduced to one term, as cx^2; also, those containing x can be reduced to one term, as dx; and all containing the known quantities can be reduced to one term, as g. Then, $cx^2 + dx = g$.

Dividing both members by c, and substituting a for the quotient of $d \div c$, and b for the quotient of $g \div c$, we have,

$$x^2 + ax = b.$$

Take any numerical quadratic, as $\frac{8x^2}{6} - \frac{2x}{3} - 8 = x^2 + \frac{4x}{6} - 4.$

Clearing of fractions, $8x^2 - 4x - 48 = 6x^2 + 4x - 24$

Transposing, etc., $2x^2 - 8x = 24$

Removing the coefficient, $x^2 - 4x = 12$

Substituting a for 4, and b for 12 in the last equation, we have,

$x^2 - ax = b$. Hence,

All affected quadratics may be reduced to the general form,

$$x^2 \pm ax = b.$$

331. The ***First Member*** of the general form of an affected quadratic equation, it will be seen, is a *Binomial*, but not a *Complete Square*. One term is *wanting* to make the square complete. (Art. 266, *note.*) The equation, therefore, cannot be solved in its present state.

329. What is an affected quadratic equation? 330. To what general form may every affected quadratic be reduced? 331. What is true of the first member of the general form of an affected quadratic?

* *Quadratic,* from the Latin *quadrare*, to make *square*.

Affected, made up of *different powers;* from the Latin *ad* and *facio*, to *make* or *join* to.

332. There are *three* methods of completing the square and solving the equation.

FIRST METHOD.

1. Given $x^2 + 2ax = b$, to find the value of x.

OPERATION.

$$x^2 + 2ax = b$$
$$x^2 + 2ax + a^2 = a^2 + b$$
$$x + a = \pm \sqrt{a^2 + b}$$
$$x = -a \pm \sqrt{a^2 + b}$$

ANALYSIS.—The *first* and *third* terms of the square of a binomial are *complete* powers, and the *second* term is twice the *product* of their *roots*; or the product of one of the roots into twice the other. (Art. 101.)

In the expression, $x^2 + 2ax$, the first term is a perfect square, and the second term $2ax$ consists of the factors $2a$ and x. But x is the root of the first term x^2; therefore, the other factor $2a$ must be *twice* the root of the third term which is required to complete the square. Hence, half of $2a$, or a, must be the root of the third term, and a^2 the term itself. Therefore, $x^2 + 2ax + a^2$ is the square of the first member completed.

But since we have added a^2 to the *first* member of the equation, we must also add it to the *second*, to preserve the equality. Extracting the square root of both members, and transposing a, we have $x = -a \pm \sqrt{a^2 + b}$, the value sought. (Art. 296.)

2. What is the value of x in $2x^2 + x = 64 - 7x$?

OPERATION.

$$2x^2 + x = 64 - 7x$$
$$2x^2 + 8x = 64$$
$$x^2 + 4x = 32$$
$$x^2 + 4x + 4 = 36$$
$$x + 2 = \pm 6$$
$$x = -2 \pm 6$$
$$i.\ e.,\ x = 4 \text{ or } -8$$

ANALYSIS. — Transposing $-7x$ and removing the coefficient of x^2, we have the form $x^2 + 4x = 32$. But the first member, $x^2 + 4x$, is an incomplete square of a binomial.

In order to complete the square, we add to it the square of half the coefficient of x. (Art. 266.) Now, having added 4 to one member of the equation, we must also add 4 to the other, to preserve their equality. Extracting the root of both, and transposing, we have $x = 4$, or -8. (Art. 297.)

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332. How many methods of completing the square?

NOTES.—1. Adding the *square* of *half the coefficient* of the second term to both members of the equation is called *completing the square.*

2. The first member of the fourth equation is the *square* of a binomial; therefore, its root is found by taking the roots of the *first* and *third* terms, which are perfect powers. (Art. 297.) From the process of squaring a binomial, it is obvious that the middle term ($4x$) forms no part of the root. (Art. 266.)

333. From these illustrations we derive the following

RULE.—I. *Reduce the equation to the form,* $x^2 \pm ax = b$.

II. *Add to each member the square of half the coefficient of* x.

III. *Extract the square root of each, and reduce the resulting equation.*

3. Find the value of x in $-2x^2 + 8ax = -6b$.

SOLUTION.—By the problem, $-2x^2+8ax = -6b$

Removing coefficient of x^2, $-x^2+4ax = -3b$

Making x^2 positive (Art, 140, Prin. 3), $x^2-4ax = 3b$

Completing square, $x^2-4ax+4a^2 = 4a^2+3b$

Extracting the root, $x-2a = \pm\sqrt{4a^2+3b}$

$\therefore\ x = 2a \pm \sqrt{4a^2+3b}$

4. Given $x^2 + ax + bx = d$, to find x.

OPERATION.

$$x^2 + ax + bx = d$$
$$x^2 + (a + b)x = d$$
$$x^2 + (a + b)x + \left(\frac{a+b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2 + d$$
$$x + \frac{a+b}{2} = \pm\sqrt{\left(\frac{a+b}{2}\right)^2 + d}$$
$$\therefore\ x = -\frac{a+b}{2} \pm \sqrt{\left(\frac{a+b}{2}\right)^2 + d}$$

ANALYSIS.—Factoring the terms which contain the first power of x, we have $ax+bx = (a+b)x$; hence, $(a+b)$ may be considered a compound coefficient of x. By adding the square of half this coefficient to both members, and extracting the root, the value of x is found.

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333. What is the rule for the first method of solving affected quadratics?

5. Given $3x - 2x^2 = -9$, to find x.

SOLUTION.—By the problem, $3x - 2x^2 = -9$

Making x^2 positive, etc., $x^2 - \frac{3x}{2} = \frac{9}{2}$

Completing square, $x^2 - \frac{3x}{2} + \frac{9}{16} = \frac{9}{2} + \frac{9}{16} = \frac{81}{16}$

Extracting root, $x - \frac{3}{4} = \pm\frac{9}{4}$

$\therefore x = \frac{3}{4} \pm \frac{9}{4}$, *i.e.*, $x = 3$ or $-1\frac{1}{2}$.

6. Given $3x^2 - 24x = -36$, to find x.

Ans. $+6$ or $+2$.

NOTE.—The two roots of an affected quadratic may have the *same* or *different* signs. Thus, in the 6th and 12th examples they are the same; in the 1st, 2d, 3d, 4th, and 5th, they are different.

7. Given $5x^2 - 40x = 45$, to find x.

8. Given $x^2 - 6ax = d$, to find x.

9. Given $2x^2 + 2ax = 2(b + c)$, to find x.

SOLUTION.—Completing the square, $x^2 + ax + \frac{a^2}{4} = \frac{a^2}{4} + b + c$

Extracting root, $x + \frac{a}{2} = \pm\sqrt{\frac{a^2}{4} + b + c}$

Transposing, $x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b + c}$

10. Given $2x^2 - 22x = 120$, to find x.

11. Given $x^2 - 140 = 13x$, to find x.

12. Find the value of x in $x^2 - 3x + 1 = 5x - 15$.

SOLUTION.—By the problem, $x^2 - 3x + 1 = 5x - 15$

Transposing, $x^2 - 8x = -16$

Completing square, $x^2 - 8x + 16 = 0$

Extracting root, $x - 4 = 0$

$\therefore x = 4$

NOTE.—In this equation, both the *signs* and the *numerical values* of the two roots are *alike*. Such equations are said to have *equal roots*.

Note.—What signs have the roots of an affected quadratic?

SECOND METHOD.

334. When an affected quadratic equation has been reduced to the general form,

$$x^2 + ax = b,$$

its root may be obtained *without recourse* to completing the square.

1. Given $x^2 + 8x = 65$, to find x.

ANALYSIS.—After the square of an affected quadratic is completed and the root extracted, the root of the third term is transposed to the second member, by changing its sign. (Art. 204.)

OPERATION.

$$x^2 + 8x = 65$$
$$x = -4 \pm \sqrt{65 + 16}$$
$$\therefore x = -4 \pm 9$$
$$i.e.,\ x = 5 \text{ or } -13$$

Now, if we prefix half the coefficient of x, with its sign changed, to plus or minus the square root of the second member increased by the square of half the coefficient of x, the second member of the equation will contain the *same combinations* of the *same terms*, as when the square is completed in the ordinary way. Hence, the

RULE.—*Prefix half the coefficient of x, with the opposite sign, to plus or minus the square root of the second member, increased by the square of half the coefficient of x.*

Solve the following equations:

2. $3x^2 - 9x - 3 = 207.$
3. $4x^2 + 12x + 5 = 45.$
4. $3x^2 - 14x + 15 = 0.$
5. $4x^2 - 9x = 28.$
6. $\dfrac{x+2}{2x} + \dfrac{2x}{x+2} = 2.$
7. $x^2 + \dfrac{ax}{b} - ab = d.$
8. $x^2 + 4ax = b.$
9. $3x^2 - 74 = 6x + 31.$
10. $x^2 + 13 = 6x.$
11. $(x-2)(x-1) = 20.$
12. $\dfrac{x+1}{x} + \dfrac{x}{x+1} = \dfrac{13}{6}.$
13. $x^2 - \dfrac{bx}{c} + ch = bd.$

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334. What is the second method of solving affected quadratics?

THIRD METHOD.

335. A third method of reducing an affected quadratic equation may be illustrated in the following manner:

1. Given $ax^2 + bx = c$, to find x.

OPERATION.

$$ax^2 + bx = c$$
$$4a^2x^2 + 4abx = 4ac$$
$$4a^2x^2 + 4abx + b^2 = 4ac + b^2$$
$$2ax + b = \pm\sqrt{4ac + b^2}$$
$$\therefore x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}$$

ANALYSIS.— Multiplying the given equation by a, the coefficient of x^2, and by 4, the smallest square number, we have

$$4a^2x^2 + 4abx = 4ac,$$

the first term of which is an exact square, whose root is $2ax$. Factoring the second term, we have $4abx = 2(2ax \times b)$. (Art. 119.)

As the factor $2ax$ is the square root of $4a^2x^2$, it is evident that $4a^2x^2$ may be regarded as the first term, and $4abx$ the middle term of the square of a binomial. Since $4abx$ is twice the product of this root $2ax$ into b, it follows that b is the second term of the binomial; consequently, b^2 added to both members will make the first a complete square, and preserve the equality. (Axiom 2.) Extracting the square root, transposing, etc., we have,

$$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}, \text{ the value of } x \text{ required.}$$

2. Given $2x^2 + 3x = 27$, to find x.

SOLUTION.—By the problem, $2x^2 + 3x = 27$

Multiplying by 4 times coef. of x^2, $16x^2 + 24x = 216$

Adding square of 3, coef. of x, $16x^2 + 24x + 9 = 225$

Extracting root, $4x + 3 = \pm 15$

Transposing, $4x = -3 \pm 15$

$\therefore x = 3$ or $-4\frac{1}{2}$.

336. From the preceding illustrations, we derive the

RULE.—I. *Reduce the equation to the form,* $ax^2 \pm bx = c$.

II. *Multiply both members by* 4 *times the coefficient of* x^2.

III. *Add the square of the coefficient of x to each member, extract the root, and reduce the resulting equation.*

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336. What is the rule for the third method of reducing affected quadratics?

NOTES.—1. When the coefficient of x is an *even* number, it is sufficient to multiply both members by the coefficient of x^2, and add to each the *square* of *half* the coefficient of x.

2. The *object* of multiplying the equation by the coefficient of x^2 is to make the first term a *perfect square* without removing the coefficient. (Art. 251.)

3. The *reason* for multiplying by 4, is that it *avoids fractions* in completing the square, when the coefficient of x is an odd number. For, multiplying both members by 4, and adding the square of the entire coefficient of x to each, is the same in effect as adding the *square of half* the coefficient of x to each, and then clearing the equation of fractions by multiplying it by the denominator 4.

4. This method of completing the square is ascribed to the Hindoos.

3. Given $3x^2 + 4x = 39$, to find x. *Ans.* 3 or $-4\frac{1}{3}$.

Reduce the following equations:

4. $x^2 - 30 = -x$.	8. $2x^2 - 6x = 8$.
5. $5x + 3x^2 = 2$.	9. $3x^2 + 5x = 42$.
6. $4x^2 - 7x - 2 = 0$.	10. $x^2 - 15x = -54$.
7. $5x^2 + 2x = 88$.	11. $9x^2 - 7x = 116$.

337. The preceding methods are equally applicable to all classes of affected quadratics, but each has its *advantages* in particular problems.

The *first* is perhaps the *most natural*, being derived from the square of a binomial; but it necessarily *involves fractions*, when the coefficient of x is an *odd number*.

The *second* is the shortest, and is therefore a *favorite* with experts in algebra.

The advantage of the *third* is, that it always *avoids fractions* in completing the square.

☞ The student should exercise his judgment as to the method best adapted to his purpose.

Notes. When the coefficient of x is an even number, how proceed? Object of multiplying by coefficient of x^2? By 4?

EXAMPLES.

Find the value of x in the following equations:

1. $x^2 - 4x = -3$.
2. $x^2 - 5x = -4$.
3. $2x^2 - 7x = -3$.
4. $x^2 + 10x = 24$.
5. $6x^2 - 13x + 6 = 0$.
6. $14x - x^2 = 33$.
7. $x^2 - 3 = \frac{x-3}{6}$.
8. $\frac{5x^2}{7} + \frac{7x}{5} = -\frac{73}{140}$.
9. $\frac{16}{x} - \frac{100 - 9x}{4x^2} = 3$.
10. $\frac{a}{x} + \frac{x}{a} = \frac{2}{a}$.
11. $x^2 + 2mx = b^2$.
12. $x^2 + \frac{x}{8} = 1\frac{1}{8}$.
13. $\frac{x^3 - 10x^2 + 1}{x^2 - 6x + 9} = x - 3$.
14. $\frac{4x}{14 - x} - \frac{x - 1}{3x} = \frac{9x + 7}{x}$.
15. $2\sqrt{x^2 - 4x} - 1 = -4x$.
16. $\sqrt{x + 5} + 6 = x + 5$.
17. $3x^2 - 7x - 20 = 0$.
18. $7x^2 - 160 = 3x$.
19. $2x^2 - 2x = 1\frac{1}{2}$.
20. $(x - 2)(x - 1) = 6$.
21. $4(x^2 - 1) = 4x - 1$.
22. $(2x - 3)^2 = 8x$.
23. $3x - 2 = \frac{14}{x - 1}$.
24. $4x - \frac{14 - x}{x + 1} = 14$.
25. $x^2 + \frac{3}{25} = \frac{4x}{5}$.
26. $x^2 + \frac{x}{2} = \frac{1}{2}$.
27. $x^2 - 2nx = m^2 - n^2$.
28. $\frac{9(b - a)}{x} = \frac{x - 3a}{b}$.

338. An Equation which contains but *two powers* of the unknown quantity, the *index* of one power being *twice* that of the other, is said to have the *Quadratic Form.*

The indices of these powers may be either *integral* or *fractional.*

Thus, $x^4 - x^2 = 12$; $x^{2n} + x^n = h$; and $\sqrt[2]{x} - \sqrt[4]{x} = c$, are equations of the quadratic form.

NOTE.—Equations of this character are sometimes called *trinomial* equations.

338. When has an equation the quadratic form? *Note.* What are such equations called?

339. Equations of the *quadratic form* may be solved by the rules for affected quadratics.

1. Given $x^4 - 2x^2 = 8$, to find x.

Solution.—By the problem, $x^4 - 2x^2 = 8$

Completing square, $x^4 - 2x^2 + 1 = 9$

Extracting square root, $x^2 - 1 = \pm 3$

Transposing, $x^2 = 4$ or -2

Extracting square root again, $x = \pm 2$, or $\pm \sqrt{-2}$

2. Given $x^6 - 4x^3 = 32$, to find x.

Solution.—By the problem, $x^6 - 4x^3 = 32$

Completing square, $x^6 - 4x^3 + 4 = 36$

Extracting square root, $x^3 - 2 = \pm 6$

Transposing, etc., $x^3 = 8$ or -4

Extracting cube root, $x = 2$ or $\sqrt[3]{-4}$

3. Given $x^{2n} - 4bx^n = a$, to find x.

Solution.—By the problem, $x^{2n} - 4bx^n = a$

Completing square, $x^{2n} - 4bx^n + 4b^2 = a + 4b^2$

Extracting square root, $x^n - 2b = \pm \sqrt{a + 4b^2}$

Transposing, $x^n = 2b \pm \sqrt{a + 4b^2}$

Extracting the nth root, $x = \sqrt[n]{2b \pm \sqrt{a + 4b^2}}$

4. Given $x^4 + 8 = 6x^2$, to find x.
5. Given $x^4 - 2x^2 = 3$, to find x.
6. Given $x^6 - 7x^3 = 0$, to find x.
7. Given $\frac{x^2}{2} + \frac{x}{4} = \frac{3}{32}$, to find x.
8. Given $\sqrt[3]{x^2} + \frac{3}{2}\sqrt[3]{x} = 1$, to find x.
9. Given $4x + 4\sqrt{x + 2} = 7$, to find x.
10. Given $\frac{\sqrt{4x + 20}}{4 + \sqrt{x}} = \frac{4 - \sqrt{x}}{\sqrt{x}}$, to find x.

PROBLEMS.

1. Find two numbers such that their sum is 12 and their product is 32.

2. A gentleman sold a picture for $24, and the per cent lost was expressed by the cost of the picture. Find the cost.

NOTE.—Let $x =$ the cost.

Then $\frac{x}{100} =$ the per cent.

We now have $x - x \times \frac{x}{100} = 24$, to find the value of x.

3. The sum of two numbers is 10 and their product is 24. What are the numbers?

4. A person bought a flock of sheep for $80; if he had purchased 4 more for the same sum, each sheep would have cost $1 less. Find the number of sheep and the price of each.

5. Twice the square of a certain number is equal to 65 diminished by triple the number itself. Required the number.

6. A teacher divides 144 oranges equally among her scholars; if there had been 2 more pupils, each would have received one orange less. Required the number in the school.

7. A father divides $50 between his two daughters, in such a proportion that the product of their shares is $600. What did each receive?

8. Find two numbers whose sum is 100 and their product 2400.

9. The fence enclosing a rectangular field is 128 rods long, and the area of the field is 1008 square rods. What are its length and breadth?

10. A colonel arranges his regiment of 1600 men in a solid body, so that each rank exceeds the file by 60 soldiers. How many does he place in rank and file?

11. A drover buys a number of lambs for \$50 and sells them at \$5.50 each, and thus gains the cost of one lamb. Required the number of lambs.

12. The sum of two numbers is 4 and the sum of their reciprocals is 1. What are the numbers?

13. The sum of two numbers is 5 and the sum of their cubes 65. What are the numbers?

14. The length of a lot is 1 yard longer than the width and the area is 3 acres. Find the length of the sides.

15. A and B start together for a place 300 miles distant; A goes 1 mile an hour faster than B, and arrives at his journey's end 10 hours before him. Find the rate per hour at which each travels.

16. A and B distribute \$1200 each among a certain number of persons. A relieves 40 persons more than B, and B gives to each person \$5 more than A. Required the number relieved by each.

17. Divide 48 into two such parts that their product may be 252.

18. Two girls, A and B, bought 10 lemons for 24 cents, each spending 12 cents; if A paid 1 cent more apiece than B, how many lemons did each buy?

19. Find the length and breadth of a room the perimeter of which is 48 feet, the area of the floor being as many square feet as 35 times the difference between the length and breadth.

20. In a peach orchard of 180 trees there are three more in a row than there are rows. How many rows are there, and how many trees in each?

21. Find the number consisting of two digits whose sum is 7, and the sum of their squares is 29.

22. The expenses of a picnic amount to \$10, and this sum could be raised if each person in the party should give 30 cts. more than the number in the party. How many compose the party?

23. Find two numbers the product of which is 120, and if 2 be added to the less and 3 subtracted from the greater, the product of the sum and remainder will also be 120.

24. Divide 36 into two such parts that their product shall be 80 times their difference.

25. The sum of two numbers is 75 and their product is to the sum of their squares as 2 to 5. Find the numbers.

26. Divide 146 into two such parts that the difference of their square roots may be 6.

27. The fore-wheel of a carriage makes sixty revolutions more than the hind-wheel in going 3600 feet; but if the circumference of each wheel were increased by three feet, it would make only forty revolutions more than the hind-wheel in passing over the same distance. What is the circumference of each wheel?

28. Find two numbers whose difference is 16 and their product 36.

29. What two numbers are those whose sum is $1\frac{1}{3}$ and the sum of their reciprocals $3\frac{1}{5}$?

30. Find two numbers whose difference is 15, and half their product is equal to the cube of the less number.

31. A lady being asked her age, said, If you add the square root of my age to half of it, and subtract 12, the remainder is nothing. What is her age?

32. The perimeter of a field is 96 rods, and its area is equal to 70 times the difference of its length and breadth. What are its dimensions?

33. The product of the ages of A and B is 120 years. If A were 3 years younger and B 2 years older, the product of their ages would still be 120. How old is each?

34. A man bought 80 pounds of pepper, and 36 pounds of saffron, so that for 8 crowns he had 14 pounds of pepper more than of saffron for 26 crowns; and the amount he laid out was 188 crowns. How many pounds of pepper did he buy for 8 crowns?

SIMULTANEOUS QUADRATIC EQUATIONS.

TWO UNKNOWN QUANTITIES.

340. A ***Homogeneous Equation*** is one in which the *sum* of the *exponents* of the unknown quantities is the same in every term which contains them.

Thus, $x^2 - y^2 = 7$, and $x^2 - xy + y^2 = 13$, are each homogeneous.

341. A ***Symmetrical Equation*** is one in which the unknown quantities are involved to the same degree.

Thus, $x^2 + y^2 = 34$, and $x^2y - xy^2 = 34$, are each symmetrical.

342. ***Simultaneous Quadratic Equations*** containing two unknown quantities, in general involve the principles of *Biquadratic* equations, which belong to the higher departments of Algebra.

There are three classes of examples, however, which may be solved by the *rules of quadratics*.

1st. When one equation is *quadratic*, and the other *simple*.

2d. When both equations are *quadratic* and *homogeneous*.

3d. When each equation is *symmetrical*.

343. To Solve Simultaneous Equations consisting of a Quadratic and a Simple Equation.

1. Given $x^2 + y^2 = 13$, and $x + y = 5$, to find x and y.

SOLUTION.—By the problem, $x^2+y^2 = 13$ (1)

" " $x+y = 5$ (2)

By transposition, $x = 5-y$ (3)

Squaring each side of (3) (Art. 102), $x^2 = 25-10y+y^2$ (4)

Substituting (4) in (1), $25-10y+y^2+y^2 = 13$ (5)

Uniting and transposing, $2y^2-10y = -12$ (6)

Comp. sq. (Art. 336, *note*), $4y^2-20y+25 = -24+25$ (7)

Extracting root, $2y-5 = \pm 1$

$\therefore\ y = 3$ or 2.

Substituting value of y in (3), $x = 2$ or 3. Hence. the

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340. What is a homogeneous equation? 341. A symmetrical equation?

RULE.—*Find the value of one of the unknown quantities in the simple equation by transposition, and substitute this value in the quadratic equation.* (Arts. 221, 223.)

Solve the following equations:

2. $x^2 + y^2 = 25$,
$x + y = 7$.

3. $x^2 + y^2 = 74$,
$x + y = 12$.

4. $x^2 - y^2 = 28$,
$x - y = 2$.

5. $x^2 + y^2 = 244$,
$y - x = 2$.

6. $3x^2 - y^2 = 251$,
$x + 4y = 38$.

7. $8x^2 + 5y^2 = 728$,
$6y - x = 15$.

344. To Solve Simultaneous Equations which are both Quadratic and Homogeneous.

8. Given $x^2+xy = 40$, and $y^2+xy = 24$, to find x and y.

SOLUTION.—By the problem, $x^2+xy = 40$ (1)

" " $y^2+xy = 24$ (2)

Let $x = py$ (3)

Substituting py in (1), $p^2y^2+py^2 = 40$ (4)

" " (2), $y^2+py^2 = 24$ (5)

Factoring, etc., (4), $y^2 = \frac{40}{p^2+p}$ (6)

" " (5), $y^2 = \frac{24}{1+p}$ (7)

Equating (6) and (7), $\frac{40}{p^2+p} = \frac{24}{1+p}$ (8)

Clearing of fractions, $5+5p = 3p^2+3p$ (9)

Transposing, etc., $3p^2-2p = 5$ (10)

Comp. sq., 3d meth. (Art. 336), $9p^2-6p+1 = 15+1=16$ (11)

Extracting root, $3p-1 = \pm 4$ (12)

Transposing, $3p = 1 \pm 4$

Dropping the negative value, $p = \frac{5}{3}$

Substituting value of p in (7), $y^2 = 24\div(1+\frac{5}{3})=9$

Extracting root, $y = \pm 3$

Substituting value of p and y in (3), $x = \frac{5}{3} \times \pm 3 = \pm 5$.

Hence, the

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343. Rule for solution of equations consisting of a quadratic and simple equation?

RULE.—I. *For one of the unknown quantities substitute the product of the other into an auxiliary quantity, and then find the value of this auxiliary quantity.*

II. *Find the values of the unknown quantities by substituting the value of the auxiliary quantity in one of the equations least involved.*

NOTE.—An auxiliary quantity is one introduced to aid in the solution of a problem, as p in the above operation.

9. Given $\left.\begin{array}{l} x + y = 9, \\ x^3 + y^3 = 189, \end{array}\right\}$ to find x and y.

10. Given $\left.\begin{array}{l} x - y = 2, \\ x^3 - y^3 = 98, \end{array}\right\}$ to find x and y.

11. Given $\left.\begin{array}{l} 3x^2 - 7y^2 = -1, \\ 4xy = 24, \end{array}\right\}$ to find x and y.

12. Given $\left.\begin{array}{l} x^2 - xy + y^2 = 19, \\ xy = 15, \end{array}\right\}$ to find x and y.

345. To Solve Simultaneous Quadratic Equations when each Equation is Symmetrical.

13. Given $x + y = 9$, and $xy = 20$, to find x and y.

SOLUTION.—By the problem, $x+y = 9$ (1)

" " $xy = 20$ (2)

Squaring (1), $x^2 + 2xy + y^2 = 81$ (3)

Multiplying (2) by 4, $4xy = 80$ (4)

Subtracting (4) from (3), $x^2 - 2xy + y^2 = 1$ (5)

Extracting sq. root of (5), $x - y = \pm 1$ (6)

Bringing down (1), $x + y = 9$

Adding (1) and (6), $2x = 10$ or 8 (7)

Removing coefficient, $x = 5$ or 4

Substituting value of x in (1), $y = 4$ or 5

NOTES.—1. The values of x and y in these equations are not equal, but interchangeable; thus, when $x = 5$, $y = 4$; and when $x = 4$, $y = 5$.

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344. How solve equations which are both quadratic and homogeneous? *Note.* What is an auxiliary quantity?

2. The solution of this class of problems *varies* according to the given equations. Consequently, no specific rules can be given that will meet every case. But judgment and practice will readily supply expedients. Thus,

I. When the sum and product are given. (Ex. 13, 15.)
Find the difference and combine it with the sum. (Art. 224.)

II. When the difference and product are given. (Ex. 16.)
Find the sum and combine it with the difference.

III. When the sum and difference of the same powers are given. (Ex. 14, 17.)
Combine the two equations by addition and subtraction.

IV. When the members of one equation are multiples of the other. (Ex. 18.)
Divide one by the other, and then reduce the resulting equation.

14. Given $\left.\begin{array}{ll} x^{\frac{1}{3}} + y^{\frac{1}{3}} = 5, & (1) \\ \text{And } x^{\frac{1}{3}} - y^{\frac{1}{3}} = 1, & (2) \end{array}\right\}$ to find x and y.

SOLUTION.—Adding (1) and (2), and dividing, $x^{\frac{1}{3}} = 3$
Involving, $x = 27$
Subtracting (2) from (1), etc., $y^{\frac{1}{3}} = 2$
Involving, $y = 8$

15. Given $\left.\begin{array}{r} x + y = 27, \\ \text{And } xy = 180, \end{array}\right\}$ to find x and y.

16. Given $\left.\begin{array}{r} x - y = 14, \\ \text{And } xy = 147, \end{array}\right\}$ to find x and y.

17. Given $\left.\begin{array}{r} x^{\frac{1}{4}} + y^{\frac{1}{4}} = 7, \\ \text{And } x^{\frac{1}{4}} - y^{\frac{1}{4}} = 3, \end{array}\right\}$ to find x and y.

18. Given $\left.\begin{array}{r} x^3y^2 + x^2y^3 = 12, \\ \text{And } x^2y + xy^2 = 6, \end{array}\right\}$ to find x and y.

NOTE.—What is true of the solution of simultaneous quadratics? When the sum and product are given, how proceed? When the difference and product? When the sum and difference of the same powers are given? When the members of one equation are multiples of the other?

PROBLEMS.

1. The difference of two numbers is 4, and the difference of their cubes 448. What are the numbers?

2. A man is one year older than his wife, and the product of their respective ages is 930. What is the age of each?

3. Required two numbers whose sum multiplied by the greater is 180, and whose difference multiplied by the less is 16.

4. In an orchard of 1000 trees, the number of rows exceeds the number of trees in each row by 15. Required the number of rows and the number of trees in each row.

5. The area of a rectangular garden is 960 square yards, and the length exceeds the breadth by 16 yards. Required the dimensions.

6. Subtract the sum of two numbers from the sum of their squares, and the remainder is 78; the product of the numbers increased by their sum is 39. What are the numbers?

7. Find two numbers whose sum added to the sum of their squares is 188, and whose product is 77.

8. A surveyor lays out a piece of land in a rectangular form, so that its perimeter is 100 rods, and its area 589 square rods. Find the length and breadth.

9. Required two numbers whose product is 28, and the sum of their squares 65.

10. A regiment of soldiers consisting of 1154 men is formed into two squares, one of which has 2 more men on a side than the other. How many men are on a side of each of the squares?

11. Required two numbers whose product is 3 times their sum, and the sum of their squares 160.

12. What two numbers are those whose product is 6 times their difference, and the sum of their squares 13?

CHAPTER XVII.

RATIO AND PROPORTION.

346. ***Ratio*** is the relation which one quantity bears to another with respect to magnitude.

347. The ***Terms of a Ratio*** are the quantities compared. The first is called the *Antecedent*, the second the *Consequent*,* and the *two* together, a *Couplet*.

348. The ***Sign*** of ratio is a *colon* (:) † placed between the two quantities compared.

Ratio is also denoted by placing the *consequent under* the *antecedent*, in the form of a *fraction*.

Thus, the ratio of a to b is written, $a : b$, or $\frac{a}{b}$.

349. The ***Measure*** or ***Value*** of a ratio is the *quotient* of the antecedent divided by the consequent, and is equal to the value of the *fraction* by which it is expressed.

Thus, the measure or value of $8 : 4$ is $8 \div 4 = 2$.

NOTE.—That quantities may have a *ratio* to each other, they must be so far of the same nature, that one can properly be said to be *equal* to, or *greater*, or *less* than the other.

Thus, a foot has a ratio to a *yard*, but not to an *hour*, or a *pound*.

350. A ***Simple Ratio*** is one which has but *two* terms; as, $a : b$, $8 : 4$.

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346. What is ratio? 347. What are the terms of a ratio? 348. The sign? How also is ratio denoted? 349. The measure or value? *Note.* What quantities have a ratio to each other? 350. What is a simple ratio?

* *Antecedent*, Latin *ante*, before, and *cedere*, to go, to *precede*.

Consequent, Latin *con*, and *sequi*, to *follow*.

† The sign of ratio (:) is derived from the sign of division (÷), the horizontal line being dropped.

351. A ***Compound Ratio*** is the product of *two or more simple* ratios.

Thus, $\left.\begin{array}{l}4:2\\9:3\end{array}\right\}$ are each simple ratios. But $4\times9:2\times3$ is a compound ratio.

NOTE.—The *nature* of compound ratios is the same as that of simple ratios. They are so called to denote their *origin*, and are usually expressed by writing the corresponding terms of the simple ratios one under another, as above.

352. A ***Direct Ratio*** arises from dividing the antecedent by the consequent.

353. An ***Inverse**** or ***Reciprocal Ratio*** arises from dividing the consequent by the antecedent, and is the same as the ratio of the *reciprocals* of the two numbers compared.

Thus, the direct ratio of a to $ab = \frac{a}{ab}$, or $\frac{1}{b}$, and that of 4 to $12 = \frac{4}{12}$, or $\frac{1}{3}$.

The inverse ratio of a to $ab = \frac{ab}{a}$, or b; of 4 to $12 = \frac{12}{4}$, or 3. It is the same as the ratio of the reciprocals, $\frac{1}{a}$ to $\frac{1}{ab}$, and $\frac{1}{4}$ to $\frac{1}{12}$.

NOTE.—A *reciprocal* ratio is expressed by inverting the *fraction* which expresses the *direct* ratio. When the *colon* is used, it is expressed by *inverting* the *order* of the terms.

354. The ratio between *two fractions* which have a common denominator, is the same as the *ratio* of their *numerators.*

Thus, the ratio of $\frac{6}{9}:\frac{3}{9}$ is the same as $6:3$.

NOTE.—When the fractions have *different denominators*, reduce them to a *common* denominator; then compare their numerators. (Art. 175.)

355. A ***Ratio of Equality*** is one in which the quantities compared are *equal*, and its *value* is a unit or 1.

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351. What is a compound ratio? *Note.* Why so called? 352. What is a direct ratio? 353. A reciprocal? 355. What is a ratio of equality?

* *Inverse*, from the Latin *in* and *verto*, to *turn upside down*, to *invert*.

356. A ***Ratio of Greater Inequality*** is one whose antecedent is *greater* than its consequent, and its value is *greater* than 1.

357. A ***Ratio of Less Inequality*** is one whose antecedent is *less* than its consequent, and its value is *less* than 1.

358. A ***Duplicate Ratio*** is the *square* of a *simple* ratio. It arises from multiplying a simple ratio into itself, or into another *equal* ratio.

359. A ***Triplicate Ratio*** is the *cube* of a *simple* ratio, and is the product of *three equal* ratios.

Thus, the *duplicate* ratio of a to b is $a^2 : b^2$.
The *triplicate* ratio of a to b is $a^3 : b^3$.

360. A ***Subduplicate Ratio*** is the *square root* of a simple ratio.

361. A ***Subtriplicate Ratio*** is the *cube root* of a simple ratio.

Thus, the *subduplicate* ratio of x to y is $\sqrt{x} : \sqrt{y}$.
The *subtriplicate* ratio of x to y is $\sqrt[3]{x} : \sqrt[3]{y}$, etc.

362. Since ratio may be expressed in the form of a *fraction*, it follows that changes made in its terms have the same effect as *like changes* in the terms of a fraction. Hence, the following

PRINCIPLES.

1°. *Multiplying the antecedent, or Dividing the consequent,*	*Multiplies the ratio.*
2°. *Dividing the antecedent, or Multiplying the consequent,*	*Divides the ratio.*
3°. *Multiplying or dividing both terms by the same quantity,*	*Does not alter the value of the ratio.*

356. Of greater inequality? 357. Of less inequality? 358. A duplicate ratio? 359. Triplicate? 360. Subduplicate? 361. Subtriplicate? 362. Name Principle 1. Principle 2. Principle 3.

EXAMPLES.

1. What is the ratio of 4 yards to 4 feet?

SOLUTION. 4 yards = 12 feet; and the ratio of 12 ft. to 4 ft. is 3.

2. What is the ratio of $6x^3$ to $2x$? *Ans.* $3x$.
3. What is the ratio of 40 square rods to an acre?
4. What is the ratio of 1 pint to a gallon?
5. What is the ratio of 64 rods to a mile?
6. What is the ratio of $8a^2$ to $4a$?
7. What is the ratio of $15abc$ to $5ab$?
8. What is the ratio of $5 to 50 cents?
9. What is the ratio of 75 cents to $6?
10. What is the ratio of 35 quarts to 35 gallons?
11. What is the ratio of $2a^2$ to $4a$?
12. What is the ratio of $x^2 - y^2$ to $x + y$?
13. What is the compound ratio of $9 : 12$ and $8 : 15$?

SOLUTION. $9 \times 8 = 72$, and $12 \times 15 = 180$. Now $72 \div 180 = \frac{8}{20}$, *Ans.* Or, $9 : 12 = \frac{9}{12}$, and $8 : 15 = \frac{8}{15}$. Now $\frac{9}{12} \times \frac{8}{15} = \frac{72}{180} = \frac{8}{20}$, *Ans.*

14. What is the compound ratio of $8 : 15$ and $25 : 30$?
15. What is the compound ratio of $a : b$ and $2b : 3ax$?
16. Reduce the ratio of 9 to 45 to the lowest terms.

SOLUTION. $9 : 45 = \frac{9}{45}$, and $\frac{9}{45} = \frac{1}{5}$, *Ans.*

17. Reduce the ratio of 24 to 96 to the lowest terms.
18. Reduce the ratio of 144 to 1728 to the lowest terms.
19. What kind of ratio is 25 to 25?
20. What kind of a ratio is $ab : ab$?
21. What kind of ratio is 35 to 7?
22. What kind of ratio is 6 to 48?
23. Which is the greater, the ratio of $15 : 9$, or $38 : 19$?
24. Which is the greater, the ratio of $8 : 25$, or $\sqrt{4} : \sqrt{25}$.
25. If the antecedent of a couplet is 56, and the ratio 8, what is the consequent?
26. If the consequent of a couplet is 7, and the ratio 14, what is the antecedent?

PROPORTION.

363. ***Proportion*** is an equality of ratios.

Thus, the ratio $8 : 4 = 6 : 3$, is a proportion. That is,

Four quantities are in *proportion*, when the *first* is the *same multiple* or *part* of the *second* that the *third* is of the *fourth*.

364. The ***Sign of Proportion*** is a double colon (: :),* or the sign (=). Thus,

The equality between the ratio of a to b and c to d is expressed by

$$a : b :: c : d, \text{ or by } \frac{a}{b} = \frac{c}{d}$$

The former is read, "a is to b as c is to d;" the latter, "b is contained in a as many times as d is contained in c."

NOTE.—Each ratio is called a *couplet*, and each term a *proportional*.

365. The ***Terms*** of a proportion are the quantities compared. The *first* and *fourth* are called the *extremes*, the *second* and *third* the *means*.

366. In every proportion there must be at least *four terms;* for the equality is between *two* or *more* ratios, and each ratio has *two* terms.

367. A proportion may, however, be formed from *three* quantities, for one of the quantities may be *repeated*, so as to form *two* terms; as, $a : b :: b : c$.

NOTE.—Care should be taken not to confound *proportion* with *ratio*. In common discourse, these terms are often used indiscriminately. Thus, it is said, "The income of one man bears a greater proportion to his capital than that of another," etc. But these are loose expressions.

In a simple ratio there are but *two* terms, an antecedent and a consequent; whereas, in a proportion there must at least be *four* terms. (Arts. 350, 366.)

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363. What is proportion? 364. The sign of proportion? *Note.* What is each ratio called? 365. What are the terms of a proportion? 366. How many terms in every proportion? 367. How form a proportion from three?

* The sign (::) is derived from the sign of equality (=), the four points being the terminations of the lines.

Again, one *ratio* may be *greater* or *less* than another, but one *proportion* is neither *greater* nor *less* than another. For equality does not admit of *degrees*. In scientific investigations, this distinction should be carefully observed.

368. A ***Mean Proportional*** between two quantities is the *middle term* or *quantity repeated*, in a proportion formed from *three quantities*.

369. A ***Third Proportional*** is the *last* term of a proportion having three quantities.

Thus, in the proportion $a : b :: b : c$, b is a mean proportional, and c a third proportional.

370. A ***Direct Proportion*** is an equality between two *direct* ratios; as, $a : b :: c : d$, $3 : 6 :: 4 : 8$.

371. An ***Inverse*** or ***Reciprocal Proportion*** is an equality between a *direct* and *reciprocal* ratio; as,

$$8 : 4 :: \tfrac{1}{3} : \tfrac{1}{6}.$$

372. ***Analogous Terms*** are the antecedent and consequent of the same couplet.

373. ***Homologous Terms*** are either two antecedents or two consequents.

PROPOSITIONS.

374. A ***Proposition*** is the statement of a truth to be proved, or of an operation to be performed.

Propositions are of two kinds, *theorems* and *problems*.

375. A ***Theorem*** is something to be proved.

376. A ***Problem*** is something to be done.

377. A ***Corollary*** is a principle *inferred* from a preceding proposition.

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368. What is a mean proportional? 369. What is a third proportional? 370. A direct proportion? 371. An inverse or reciprocal proportion? 372. What are analogous terms? 373. Homologous? 374. What is a proposition? How divided? 375. What is a theorem? 376. A problem? 377. A corollary?

378. The more important theorems in proportion are the following:

Theorem I.

If four quantities are proportional, the product of the extremes is equal to the product of the means.

Let	$a : b :: c : d$
By Art. 363,	$\frac{a}{b} = \frac{c}{d}$
Clearing of fractions,	$ad = bc.$

Verification by Numbers.

Given, $2 : 4 :: 8 : 16$; and $2 \times 16 = 4 \times 8.$

Cor.—The *relation* of the four terms of a proportion to each other is such, that if *any three* of them are given, the *fourth* may be found.

Thus, since $ad = bc$, it follows that

$a = bc \div d,\ b = ad \div c,\ c = ad \div b,$ and $d = bc \div a.$ (Ax. 5.)

Notes.—1. The rule of *Simple Proportion* in Arithmetic is founded upon this principle, and its operations are easily proved by it.

2. This theorem furnishes a very simple test for determining whether any four quantities are proportional. We have only to multiply the extremes together, and the means.

Theorem II.

If three quantities are proportional, the product of the extremes is equal to the square of the mean.

Let	$a : b :: b : c$
By Art. 363,	$\frac{a}{b} = \frac{b}{c}$
Clearing of fractions,	$ac = b^2.$

Again, $9 : 6 :: 6 : 4$, and $4 \times 9 = 6^2.$

Cor.—A *mean proportional* between *two quantities* is equal to the *square root* of their product.

THEOREM III.

If the product of two quantities is equal to the product of two others, the four quantities are proportional; the factors of either product being taken for the extremes, and the factors of the other for the means.

Let $ad = bc$

Dividing by bd, $\frac{a}{b} = \frac{c}{d}$

Or, by Art. 363, $a : b :: c : d.$

Again, $4 \times 6 = 3 \times 8$, and $4 : 3 :: 8 : 6.$

THEOREM IV.

If four quantities are proportional, they are proportional when the means are inverted.

Let $a : b :: c : d$, then $a : c :: b : d$

For, by Art. 363, $\frac{a}{b} = \frac{c}{d}$

Multiplying by $\frac{b}{c}$, $\frac{a}{c} = \frac{b}{d}$

Or, $a : c :: b : d.$

Again, $3 : 6 :: 4 : 8$, and $3 : 4 :: 6 : 8$. (Th. I.)

NOTE.—This change in the order of the means is called "*Alternation*."

THEOREM V.

If four quantities are proportional, they are proportional when the terms of each couplet are inverted.

Let $a : b :: c : d$, then $b : a :: d : c$

By Theorem I, $ad = bc$

By Theorem 3, $b : a :: d : c.$

Again, $6 : 2 :: 15 : 5$, then $2 : 6 :: 5 : 15$. (Th. I.)

COR.—If the *extremes* are inverted, or the *order* of the terms, the quantities will be proportional.

NOTES.—1. If the terms of only *one* of the couplets are inverted, the proportion becomes *reciprocal.*

2. The change in the order of the terms of each couplet is called "*Inversion.*"

3. This proposition supposes the quantities compared to be of the same kind. Thus, a line has no relation to weight. (Art. 349, *note.*)

THEOREM VI.

If four quantities are proportional, two analogous or two homologous terms may be multiplied or divided by the same quantity without destroying the proportion.

Let $a : b :: c : d$

Multiplying analogous terms, $am : bm :: c : d$

and $a : b :: cm : dm$

For, $\dfrac{a}{b} = \dfrac{c}{d}$

Hence, (Art. 362, Prin. 3), $\dfrac{am}{bm} = \dfrac{c}{d}$, and $\dfrac{a}{b} = \dfrac{cm}{dm}$

Multiplying homologous terms, $am : b :: cm : d$

And $a : bm :: c : dm.$

Hence, (Ax. 4, 5), $\dfrac{am}{b} = \dfrac{cm}{d}$, and $\dfrac{a}{bm} = \dfrac{c}{dm}$

Dividing analogous terms, $\dfrac{a}{m} : \dfrac{b}{m} :: c : d,$

and $a : b :: \dfrac{c}{m} : \dfrac{d}{m}$

Dividing homologous terms, $\dfrac{a}{m} : b :: \dfrac{c}{m} : d$

and $a : \dfrac{b}{m} :: c : \dfrac{d}{m}$

Clearing of fractions (Th. 1), $ad = bc$

COR.—*All the terms* of a proportion may be multiplied or divided by the *same quantity* without *destroying* the proportion.

NOTES.—1. When the *homologous* terms are multiplied or divided, both ratios are *equally increased* or *diminished.*

2. When the *analogous* terms are multiplied or divided, the ratios are *not altered.*

THEOREM VII.

If four quantities are proportional, the sum of the first and second is to the second, as the sum of the third and fourth is to the fourth.

Let $a : b :: c : d$, then $a+b : b :: c+d : d$

For, $\frac{a}{b} = \frac{c}{d}$

Adding 1 to each member, $\frac{a}{b} + 1 = \frac{c}{d} + 1$. (Ax. 2.)

Incorporating 1, $\frac{a+b}{b} = \frac{c+d}{d}$

Therefore (Art. 363), $a+b : b :: c+d : d$

Again, $4 : 2 :: 6 : 3$, then $4+2 : 2 :: 6+3 : 3$

NOTE.—This combination is sometimes called "*Composition.*"

THEOREM VIII.

If four quantities are proportional, the difference of the first and the second is to the second, as the difference of the third and fourth is to the fourth.

Let $a : b :: c : d$, then $a-b : b :: c-d : d$

For, $\frac{a}{b} = \frac{c}{d}$

Subtracting 1 from each member, $\frac{a}{b} - 1 = \frac{c}{d} - 1$

Incorporating -1, $\frac{a-b}{b} = \frac{c-d}{d}$

Therefore, $a-b : b :: c-d : d$

Again, $4 : 2 :: 6 : 3$, then $4-2 : 2 :: 6-3 : 3$

NOTE.—This comparison is sometimes called "*Division.*" *

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* The technical terms, *Composition* and *Division*, are calculated rather to *perplex* than to *aid* the learner, and are properly falling into disuse. The objection to the former is, that it is liable to be mistaken for the *composition* or *compounding* of *ratios*, whereas the two

Theorem IX.

If two ratios are respectively equal to a third, they are equal to each other.

Let $a : b :: m : n$, and $c : d :: m : n$

Then $\frac{a}{b} = \frac{m}{n}$, and $\frac{c}{d} = \frac{m}{n}$

By Ax. 1, $\frac{a}{b} = \frac{c}{d}$. That is, $a : b = c : d$

Again, $12 : 4 = 6 : 2$, and $9 : 3 = 6 : 2$

$\therefore\ 12 : 4 = 9 : 3$

Theorem X.

When any number of quantities are proportional, any antecedent is to its consequent, as the sum of all the antecedents is to the sum of all the consequents.

Let $a : b :: c : d :: e : f$, etc.

Then $a : b :: a+c+e : b+d+f$, etc.

For (Th. 1), $ad = bc$

And, " $af = be$

Also, $ab = ba$

Adding (Ax, 2), $ab+ad+af = ba+bc+be$

Factoring, $a(b+d+f) = b(a+c+e)$

Hence, (Th. 3), $a : b :: (a+c+e$, etc.$) : (b+d+f$, etc.$)$

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operations are *entirely different.* In one the terms are *added,* in the other they are *multiplied* together. (Art. 351.)

The objection to the latter is, that the change to which the term *division* is here applied, is effected by *subtraction,* and has no reference to *division,* in the sense the word is used in Arithmetic and Algebra. Moreover, the alteration in the terms of Theorem 6 is produced by *actual division.* Usage, however ancient, can *no longer justify* the employment of the *same* word in *two different senses,* in explaining the *same* subject.

THEOREM XI.

If the corresponding terms of two or more proportions are multiplied together, the products will be proportional.

Let $a : b :: c : d$, and $e : f :: g : h$

Then $ae : bf :: cg : dh$

For, $\frac{a}{b} = \frac{c}{d}$ and $\frac{e}{f} = \frac{g}{h}$

Mult. ratios together (Ax. 4), $\frac{ae}{bf} = \frac{cg}{dh}$

Hence, (Th. 3), $ae : bf :: cg : dh$.

THEOREM XII.

If four quantities are proportional, like powers or roots of these quantities are proportional.

Let $a : b :: c : d$, then $\frac{a}{b} = \frac{c}{d}$

By Ax. 10, $\frac{a^n}{b^n} = \frac{c^n}{d^n}$

Hence (Th. 3), $a^n : b^n :: c^n : d^n$

Extracting sq. root, $a^{\frac{1}{2}} : b^{\frac{1}{2}} :: c^{\frac{1}{2}} : d^{\frac{1}{2}}$

Again, $2 : 3 :: 4 : 6$, then $2^3 : 3^3 :: 4^3 : 6^3$

In like manner, $\sqrt{4} : \sqrt{9} :: \sqrt{16} : \sqrt{36}$.

NOTE.—The index n may be either *integral* or *fractional*.

THEOREM XIII.

Equimultiples of two quantities are proportional to the quantities themselves.

Since $\frac{a}{b} = \frac{c}{d}$, by Art. 362, Prin. 3, $\frac{am}{bm} = \frac{c}{d}$

Hence, $am : bm :: c : d$,

PROBLEMS.

1. The first three terms of a proportion are 6, 8, and 3. What is the fourth?

Let $x =$ the fourth term,

Then $6 : 8 :: 3 : x$

$\therefore\ 6x = 24$, and $x = 4$.

2. The last three terms of a proportion are 8, 6, and 12. What is the first?

3. Required a third proportional to 25 and 400.

4. Required a mean proportional between 9 and 16.

5. Find two numbers, the greater of which shall be to the less, as their sum to 42; and as their difference to 6.

6. Divide the number 18 into two such parts, that the squares of those parts may be in the ratio of 25 to 16.

7. Divide the number 28 into two such parts, that the quotient of the greater divided by the less shall be to the quotient of the less divided by the greater as 32 to 18.

8. What two numbers are those whose product is 24, and the difference of their cubes is to the cube of their difference as 19 to 1?

9. Find two numbers whose sum is to their difference as 9 is to 6, and whose difference is to their product as 1 to 12.

10. A rectangular farm contains 860 acres, and its length is to its breadth as 43 to 32. What are the length and breadth?

11. There are two square fields; a side of one is 10 rods longer than a side of the other, and the areas are as 9 to 4. What is the length of their sides?

12. What two numbers are those whose product is 135, and the difference of their squares is to the square of their difference as 4 to 1?

13. Find two numbers whose product is 320; and the difference of their cubes is to the cube of their difference as 61 to 1.

CHAPTER XVIII.

PROGRESSION.

379. A ***Progression*** is a series of quantities which *increase* or *decrease* according to a fixed law.

380. The ***Terms of a Progression*** are the quantities which form the series. The *first* and *last* terms are the *extremes;* the others, the *means.*

381. Progressions are of three kinds: *arithmetical, geometrical,* and *harmonical.*

ARITHMETICAL PROGRESSION.

382. An ***Arithmetical Progression*** is a series which *increases* or *decreases* by a *constant quantity* called the *common difference.*

383. In an *ascending* series, each term is found *by adding the common difference* to the preceding term.

If the first term is a, and the common difference d, the series is

$$a,\ a+d,\ a+2d,\ a+3d, \text{ etc.}$$

If $a=2$, and $d=3$, the series is 2, 5, 8, 11, 14, etc.

384. In a *descending* series, each term is found by *subtracting* the common difference from the preceding term.

If a is the first term, and d the common difference, the series is

$$a,\ a-d,\ a-2d,\ a-3d, \text{ etc.}$$

In this case, the common difference may be considered $-d$. Hence, the common difference may be either *positive* or *negative.* And, since adding a *negative* quantity is equivalent to subtracting an equal

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379. What is a progression? 380. The terms? 381. How many kinds of progression? 382. An arithmetical progression? What is this constant quantity called? 383. An ascending series? 384. A descending series?

positive one, it may therefore properly be said that each successive term of the series is derived from the preceding by the *addition* of the *common difference.* (Art. 75, Prin. 3.)

NOTES.—1. The *common difference* was formerly called *arithmetical ratio;* but this term is passing out of use.

2. An Arithmetical Progression is sometimes called an *Equidifferent Series*, or a *Progression by Difference.* In every progression there may be an infinite number of terms.

385. *If four quantities are in arithmetical progression, the sum of the extremes is equal to the sum of the means.*

Let $a, a+d, a+2d, a+3d,$ be the series.
Adding extremes, etc., $2a+3d = 2a+3d.$

Or, let $2, 2+3, 2+6, 2+9,$ be the series.
Then $2+2+9=2+3+2+6.$

386. *If three quantities are in arithmetical progression, the sum of the extremes is equal to double the mean.*

Let $a, a+d, a+2d,$ be the series.
Then $2a+2d = 2(a+d).$

Again, let $2, 2+4, 2+8,$ be the series.
Then $2+2+8 = 2(2+4).$

COR.—An *Arithmetical Mean* between two quantities may be found by taking *half their sum.*

387. In ***Arithmetical Progression*** there are *five elements* to be considered: the *first* term, the *common difference,* the *last* term, the *number* of terms, and the *sum* of the terms.

Let a = the first term.
d = the common difference,
l = the last term.
n = the number of terms.
s = the sum of the terms.

The relation of these five quantities to each other is such that if any *three* of them are given, the *other two* can be found.

385. What is true of four quantities in arithmetical progression? 386. Of three quantities? 387. Name the elements in arithmetical progression? What relation have they to each other?

CASE I.

388. The First Term, the Common Difference, and Number of Terms being given, to Find the *Last Term*.

Each succeeding term of a progression is found by adding the common difference to the preceding term. (Art. 384.) Therefore the terms of an *ascending* series are

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \text{etc.}$$

The terms of a *descending* series are

$$a, \quad a-d, \quad a-2d, \quad a-3d, \quad \text{etc.}$$

It will be seen that the coefficient of d in each term of both series is *one less* than the number of that term in the series. Therefore, putting l for the last or nth term, we have

FORMULA I. $$l = a \pm (n - 1)\,d.$$

RULE.—I. *Multiply the common difference by the number of terms less one.*

II. *When the series is ascending, add this product to the first term; when descending, subtract it from the first term.*

1. Given $a = 3$, $d = 2$, and $n = 7$, to find l.

$$l = a \pm (n-1)\,d = 3 + (7-1)\,2 = 15, \text{ Ans.}$$

2. Given $a = 25$, $d = -2$, and $n = 9$, to find l.
3. Given $a = 12$, $d = 4$, and $n = 15$, to find l.
4. Given $a = 1$, $d = -\frac{1}{4}$, and $n = 13$, to find l.
5. Given $a = \frac{3}{4}$, $d = \frac{1}{8}$, and $n = 9$, to find l.
6. Given $a = 1$, $d = -.01$, and $n = 10$, to find l.
7. Find the 12th term of the series 3, 5, 7, 9, 11, etc.

NOTE.—In this problem, $a = 3$, $d = 2$, $n = 12$. *Ans.* 25.

8. Find the 15th term of 1, 4, 7, 10, etc.
9. Find the 9th term of 31, 29, 27, 25, etc.
10. What is the 30th term of the series 1, $2\frac{1}{2}$, 4, $5\frac{1}{2}$, etc.
11. Find the 25th term of the series $x + 3x + 5x + 7x$, etc.
12. Find the nth term of the series $2a$, $5a$, $8a$, $11a$, etc.

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388. What is the rule for finding the last term?

CASE II.

389. The Extremes and Number of Terms being given, to Find the *Sum* of the Series.

Let a, $a+d$, $a+2d$, $a+3d$ . . . l, be an arithmetical progression, the sum of which is required.

Since the sum of two or more quantities is the same in whatever order they are added (Art. 63, Prin. 2), we have

$$s = a + (a+d) + (a+2d) + (a+3d) + \ldots + l$$

Inverting, $s = l + (l-d) + (l-2d) + (l-3d) + \ldots + a$

Adding, $2s = a + l + (a+l) + (a+l) + (a+l) + \ldots + a + l$

$\therefore$ $2s = (a+l)$ taken n times, or as many times as there are terms in the series.

That is, $2s = (a+l)\,n$. Hence, the

FORMULA II. $$s = \frac{(a+l)}{2} \times n.$$

RULE.—*Multiply half the sum of the extremes by the number of terms.*

COR.—From the preceding illustration it follows that the sum of the extremes is equal to the sum of any two terms equally distant from the extremes.

Thus, in the series, 3, 5, 7, 9, 11, 13, the sum of the first and last terms, of the second and fifth, etc., is the same, viz., 16.

1. Given $a = 4$, $l = 148$, and $n = 15$, to find s.

SOLUTION. $4 + 148 = 152$, and $(152 \div 2) \times 15 = 1140$, *Ans.*

2. Given $a = \frac{1}{2}$, $l = 30$, and $n = 50$, to find s.
3. Given $a = 6$, $l = 42$, and $n = 9$, to find s.
4. Given $a = 5$, $l = 75$, and $n = 35$, to find s.
5. Given $a = 2$, $l = 1$, and $n = 17$, to find s.
6. Find the sum of the series 2, 5, 8, 11, etc., to 20 terms.
7. Find the sum of the series 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, etc., to 25 terms.
8. Find the sum of the series 75, 72, 69, 66, 63, etc., to 15 terms.

390. The two preceding formulas are fundamental, and furnish the means for solving all the problems in Arithmetical Progression. From them may be derived *eighteen other* formulas.

By Formula I.

391. This formula contains *four different* quantities; the *first* term, the *common difference*, the *last* term, and the *number* of terms. If *any three* of these quantities are given, the *other* may be found. (Art. 388.)

1. $l = a \pm (n - 1)\,d$; a, d, and n being given.

3. Given d, l, and n, to find a, the first term.*

Transposing $(n-1)\,d$ in (1),

$$a = l \pm (n-1)\,d.$$

4. Given a, l, and n, to find d, the common difference.

Transposing in (1), and dividing by $(n-1)$,

$$d = \frac{l-a}{n-1}.$$

5. Given a, d, and l, to find n, the number of terms.

Clearing of fractions and reducing (4),

$$n = \frac{l-a}{d} + 1.$$

1. Given $a = 25$, $d = 3$, and $n = 12$, to find l.
2. Given $a = 58$, $d = 5$, and $n = 45$, to find l.
3. Given $d = 3$, $l = 35$, and $n = 9$, to find a.
4. Given $l = 57$, $d = 5$, and $n = 21$, to find a.
5. Given $a = 15$, $l = 85$, and $n = 31$, to find d.
6. Given $a = 28$, $l = 7$, and $n = 26$, to find d.
7. Given $a = 23$, $d = 5$, and $l = 5138$, to find n.
8. Given $a = 6$, $d = 6$, and $l = 1152$, to find n.

* For Formula 2, see 389.

BY FORMULA II.

392. In this formula there are *four different* quantities: the *first* term, the *last* term, the *number* of terms, and the *sum* of the terms. If *any three* of these quantities are given, the other may be found. (Art. 389.)

2. $s = \frac{a + l}{2} \times n$, a, l, and n being given.

NOTE.—For Formulas 3–5, see Article 391.

6. Given l, n, and s, to find a, the first term.

Clearing (2) of fractions, dividing and transposing,

$$a = \frac{2s}{n} - l.$$

7. Given a, n, and s, to find l, the last term.

Transposing in (6), we have

$$l = \frac{2s}{n} - a.$$

8. Given a, l, and s, to find n, the number of terms.

Clearing (7) of fractions, transposing, factoring, and dividing,

$$n = \frac{2s}{a + l}.$$

1. Given $a = 9$, $l = 41$, and $n = 7$, to find s.
2. Given $a = \frac{1}{5}$, $l = 45$, and $n = 50$, to find s.
3. Given $l = 50$, $d = 4$, and $n = 12$, to find a.
4. Given $a = 9$, $l = 41$, and $s = 150$, to find n.
5. Given $d = 7$, $l = 21$, and $n = 35$, to find a.
6. Given $a = 46$, $l = 24$, and $s = 455$, to find n.
7. Given $a = 27$, $n = 9$, and $s = 72$, to find l.
8. Given $a = 72$, $n = 8$, and $s = 288$, to find l.
9. Find the sum of the series 3, 5, 7, 9, etc., to 15 terms.
10. Find the twentieth term of 5, 8, 11, 14, 17, etc.
11. If the first term of an ascending series is 5, and the common difference 4, what is the 15th term?

393. The remaining twelve formulas are derived by combining the preceding ones in such a manner as to eliminate the quantity whose value is not sought. They are contained in the following

TABLE.

No.	Given.	Required.	Formulas.
9.	d, n, s	a	$a = \dfrac{2s - dn^2 + dn}{2n}$
10.	d, l, s	a	$a = \dfrac{d}{2} \pm \sqrt{\left(l + \dfrac{d}{2}\right)^2 - 2ds}$
11.	a, l, s	d	$d = \dfrac{l^2 - a^2}{2s - l - a}$
12.	l, n, s	d	$d = \dfrac{2(nl - s)}{n(n - 1)}$
13.	a, n, s	d	$d = \dfrac{2s - 2an}{n^2 - n}$
14.	d, n, s	l	$l = \dfrac{s}{n} + \dfrac{(n - 1)d}{2}$
15.	a, d, s	l	$l = -\dfrac{d}{2} \pm \sqrt{2ds + \left(a - \dfrac{d}{2}\right)^2}$
16.	a, d, s	n	$n = \dfrac{\pm\sqrt{(2a - d)^2 + 8ds} - 2a + d}{2d}$
17.	d, l, s	n	$n = \dfrac{2l + d \pm \sqrt{(2l + d)^2 - 8ds}}{2d}$
18.	a, d, n	s	$s = \dfrac{n}{2}[2a + (n - 1)d]$
19.	a, d, l	s	$s = \dfrac{l + a}{2} + \dfrac{l^2 - a^2}{2d}$
20.	d, l, n	s	$s = \dfrac{n}{2}[2l - (n - 1)d]$

☞ Of the twenty formulas in Arithmetical Progression, the *first two* are *indispensable*, and should be *thoroughly committed* to memory; the *next six* are important in the solution of particular problems. The remaining twelve are of less consequence, but will be found interesting to the inquisitive student.

394. By the fourth formula in Art. 391, any number of arithmetical means may be inserted between two given terms of an arithmetical progression. For, the *number* of terms consists of the *two extremes* and all the *intermediate* terms.

Let m = the number of means to be inserted.
Then $m+2 = n$, the whole number of terms.
Substituting $m+2$ for n in the fourth formula, we have

$$d = \frac{l-a}{m+1}.\quad \text{Hence,}$$

The required number of means is found by the continued addition of the common difference to the successive terms.

1. Find 4 arithmetical means between 1 and 31.
2. Find 9 arithmetical means between 3 and 48.

PROBLEMS.

1. If the first term of an ascending series is 5, the common difference 3, and the number of terms 15, what is the last term?

2. If the first term of a descending series is 27, the common difference 3, and the number of terms 12, what is the last term?

3. If the first term of an ascending series be 7, and the common difference 5, what will the 20th term be?

4. Find 5 arithmetical means between 2 and 60.

5. What is the sum of 100 terms of the series $\frac{1}{3}$, $\frac{2}{3}$, 1, $\frac{4}{3}$, $\frac{5}{3}$, 2, $\frac{7}{3}$, $\frac{8}{3}$, 3, etc.

6. If the sum of an arithmetical series is 18750, the least term 5, and the number of terms 20, what is the common difference?

7. Required the sum of the odd numbers 1, 3, 5, 7, 9, 11, etc., continued to 76 terms?

8. Required the sum of 100 terms of the series of even numbers 2, 4, 6, 8, 10, etc.

9. The extremes of a series are 2 and 47, and the number of terms is 10. What is the common difference?

10. Insert 8 means between 6 and 72.

11. Insert 9 means between 12 and 108.

12. The first term of a descending series is 100, the common difference 5, and the number of terms 15. What is the sum of the terms?

NOTE.—1. In Arithmetical Progression, problems often occur in which the terms are not directly given, but are implied in the conditions. Such problems may be solved by stating the conditions algebraically, and reducing the equations.

13. Find four numbers in arithmetical progression, whose sum shall be 48, and the sum of their squares 656.

Let $x =$ the second of the four numbers.

And $y =$ their common difference.

By the conditions, $(x-y)+x+(x+y)+(x+2y) = 48$ (1)

And $(x-y)^2+x^2+(x+y)^2+(x+2y)^2 = 656$ (2)

Uniting terms in (1), $4x+2y = 48$ (3)

" " " (2), $4x^2+4xy+6y^2 = 656$ (4)

Transposing and dividing in (3), $y = 24-2x$ (5)

Dividing (4) by 2, $2x^2+2xy+3y^2 = 328$ (6)

Substituting value of y, $2x^2+2x(24-2x)+3(24-2x)^2 = 328$

Reducing, $x^2-24x = -140$

Completing square, etc., $x = 14$ or 10

Substituting in (5) $y = -4$ or 4

Hence the required numbers are 6, 10, 14, and 18.

NOTE.—2. The first two values of x and y produce a *descending* series; the other two an *ascending* series. In both the numbers are the same.

14. Find three numbers in arithmetical progression whose sum is 15, and the sum of their cubes is 495.

15. If 100 marbles are placed in a straight line a yard apart, how far must a person travel to bring them one by one to a box a yard from the first marble?

16. How many strokes does a common clock strike in 24 hours?

17. A student bought 25 books, and gave 10 cents for the first, 30 cents for the second, 50 cents for the third, etc. What did he pay for the whole?

18. A boy puts into his bank a cent the first day of the year, 2 cents the second day, 3 cents the third day, and so on to the end of the year. What sum does he thus lay up in 365 days?

19. The clocks of Venice go on to 24 o'clock. How many strokes does one of them strike in a day?

20. What will be the amount of \$1, at 6 per cent simple interest, in 20 years?

21. What three numbers are those whose sum is 120, and the sum of whose squares is 5600?

22. A traveller goes 10 miles a day; three days after, another follows him, who goes 4 miles the first day, 5 the second, 6 the third, and so on. When will he overtake the first?

23. Find four numbers, such that the sum of the squares of the extremes is 4500, and the sum of the squares of the means is 4100.

24. A sets out from a certain place and goes 1 mile the first day, 3 miles the second day, 5 the third, etc. After he has been gone 3 days, he is followed by B, who goes 11 miles the first day, 12 the second, etc. When will B overtake A?

25. The first term of a decreasing arithmetical progression is 10, the common difference $\frac{1}{3}$, and the number of terms 21. Required the sum of the series.

26. A debt can be discharged in 60 days by paying \$1 the first day, \$4 the second, \$7 the third, etc. Required the amount of the debt and of the last payment.

GEOMETRICAL PROGRESSION.

395. A ***Geometrical Progression*** is a series of quantities which *increase* or *decrease* by a *constant multiplier* called the *ratio.* Hence,

The *ratio* may be an *integer* or a *fraction.*

NOTE.—When the ratio is *fractional*, the series will decrease. For multiplying by a fraction is taking a certain part of the multiplicand as *many times* as there are *like parts* of a *unit* in the multiplier.

396. In a geometrical series, each succeeding term is found by multiplying the preceding one by the ratio.

Thus, if a is the first term, and r the ratio, the series is

$$a,\quad ar,\quad ar^2,\quad ar^3,\quad ar^4,\quad ar^5,\quad ar^6,\quad \text{etc.}$$

If the ratio is 3, the series is

$$a,\quad a\times 3,\quad a\times 3^2,\quad a\times 3^3,\quad \text{etc.}$$

If the ratio is $\frac{1}{2}$, the series is

$$a,\quad a\times\tfrac{1}{2},\quad a\times\tfrac{1}{2}\times\tfrac{1}{2},\quad a\times\tfrac{1}{2}\times\tfrac{1}{2}\times\tfrac{1}{2},\quad \text{etc.}$$

397. An ***Ascending Series*** is one which increases by an integral ratio; as, 2, 4, 8, 16, 32, etc.

398. A ***Descending Series*** is one which decreases by a fractional ratio; as, 64, 32, 16, 8, etc.

399. When the ratio is a *positive* quantity, all the terms of the progression are *positive;* when it is *negative,* the terms are alternately *positive* and *negative.*

Thus, if the first term is a, and the ratio -3, the series is

$$a,\quad -3a,\quad +9a,\quad -27a,\quad +81a,\quad \text{etc.}$$

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395. What is a geometrical progression? 397. What is an ascending series? 398. Descending? 399. What law governs the signs?

400. In geometrical progression there are five elements: the *first* term, the *last* term, the *number* of terms, the *common ratio*, and the *sum* of the terms.

Let a = the first term,
l = the last term,
n = the number of terms,
r = the ratio,
s = the sum of the terms.

The relation of these five quantities to each other is such that if *any three* of them are given, the *other two* can be found.

CASE I.

401. The First Term, the Number of Terms, and the Ratio being given, to Find the Last Term.

In this problem, a, n, and r are given, to find l, the last term.

The successive terms of the series are

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad \text{etc., to} \quad ar^{n-1}. \quad \text{(Art. 397.)}$$

By inspection, it will be seen that the ratio r consists of a regular series of powers, and in each term the index of the power is *one less* than the number of the terms. Therefore, the last or nth term of the series is ar^{n-1}. Hence, we have

FORMULA I. $\quad l = ar^{n-1}$.

RULE.—*Multiply the first term by that power of the ratio whose index is one less than the number of terms.*

COR.—Any term in a series may be found by the preceding rule; for the series may be supposed to stop at that term.

1. Given $a = 5$, $n = 6$, and $r = 2$, to find l.
2. Given $a = 2$, $n = 8$, and $r = 3$, to find l.
3. Given $a = 72$, $n = 5$, and $r = \frac{1}{2}$, to find l.
4. Given $a = 5$, $n = 4$, and $r = 4$, to find l.
5. Given $a = 7$, $n = 5$, and $r = 2$, to find l.
6. Given $a = 10$, $n = 6$, and $r = -5$, to find l.

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400. Name the elements in geometrical progression. 401. How find the last term?

CASE II.

402. The First Term, the Last Term, and the Ratio being given, to Find the *Sum* of the Terms.

In this problem, a, l, and r are given, to find s.

Since $s =$ the sum of the terms, we have

$$s = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}. \qquad (1)$$

Multiplying (1) by r,

$$rs = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n. \qquad (2)$$

Subtracting (1) from (2), $\quad rs - s = ar^n - a. \qquad (3)$

Factoring and dividing, $\quad s = \frac{ar^n - a}{r - 1}. \qquad (4)$

In equation (4), ar^n is the last term of (2), and is therefore the product of the ratio by the last term in the given series.

Substituting lr for ar^n, we have

$$\text{FORMULA II.} \qquad s = \frac{lr - a}{r - 1}.$$

RULE.—*Multiply the last term by the ratio, from the product subtract the first term, and divide the remainder by the ratio less one.*

☞ For the method of finding the sum of an *infinite descending series*, see Art. 435.

1. Given $a = 2$, $l = 500$, and $r = 3$, to find the sum.

SOLUTION. $s = \frac{lr - a}{r - 1} = \frac{500 \times 3 - 2}{2} = 749$, *Ans.*

2. Given $a = 8$, $l = 2000$, and $r = 5$, to find s.
3. Given $a = 9$, $l = 5000$, and $r = 10$, to find s.
4. Given $a = 5$, $l = 25000$, and $r = 4$, to find s.
5. Given $a = 15$, $l = 20$, and $r = 6$, to find s.
6. Given $a = 25$, $l = 12$, and $r = 4$, to find s.

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402. How find the sum of the terms?

403. The two preceding formulas furnish the means for solving all problems in geometrical progression. They may be varied so as to form *eighteen* other formulas.

BY FORMULA I.

404. The first formula contains *four different* quantities: the *first* term, the *last* term, the *ratio*, and the *number* of terms. If *any three* of these quantities are given, the *other* may be found. By the first formula,

1. $l = ar^{n-1}$; a, n, and r being given. (Art. 401.)

For formula 2, see Article 402.

3. Given l, n, and r, to find a, the first term.

Factoring (1), and dividing by r^{n-1},

$$a = \frac{l}{r^{n-1}}.$$

4. Given a, l, and n, to find r, the ratio.

Dividing (1) by a, and extracting the root denoted by the index,

$$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$$

5. Given a, l, and r, to find n, the number of terms.

Dividing (1) by a, $\quad r^{n-1} = \frac{l}{a}.$

By logarithms, $\quad \log r\,(n-1) = \log l - \log a$

Dividing, etc., $\quad n = \frac{\log l - \log a}{\log r} + 1.$

NOTE.—Since this formula contains logarithms, it may be deferred till that subject is explained.

1. Given $a = 3$, $n = 5$, and $r = 10$, to find l.
2. Given $a = 5$, $n = 6$, and $r = 5$, to find l.
3. Given $l = 256$, $n = 8$, and $r = 2$, to find a.
4. Given $l = 243$, $n = 5$, and $r = 3$, to find a.
5. Given $a = 2$, $l = 2592$, and $n = 5$, to find r.
6. Given $a = 4$, $l = 2500$, and $n = 5$, to find r.

By Formula II.

405. This formula contains four different quantities: the *first* term, the *last* term, the *ratio*, and the *sum* of the terms. If *any three* of them are given, the *other* may be found. By the second formula,

2. $s = \dfrac{lr - a}{r - 1}$, a, l, and r being given. (Art. 402.)

For formulas 3–5, see Article 404.

6. Given l, r, and s, to find a, the first term.

Clearing (2) of fractions, etc.,

$$a = lr - s(r - 1)$$

7. Given a, r, and s, to find l, the last term.

Transposing in (6),

$$lr = a + s(r - 1).$$

Dividing by r,

$$l = \frac{a + s(r - 1)}{r}.$$

8. Given a, l, and s, to find r, the ratio.

Clearing (2) of fractions,

$$sr - s = lr - a.$$

Transposing in the last equation,

$$sr - lr = s - a.$$

Factoring, etc.,

$$r = \frac{s - a}{s - l}.$$

1. Given $a = 2$, $l = 108$, and $r = 3$, to find s.
2. Given $l = 54$, $r = 3$, and $s = 80$, to find a.
3. Given $a = 4$, $r = 5$, and $s = 160$, to find l.
4. Given $a = 4$, $l = 12500$, and $s = 15624$, to find r.
5. Given $a = 5$, $l = 150$, and $r = 6$, to find s.
6. Given $a = 7$, $r = 10$, and $s = 200$, to find l.

406. The remaining twelve formulas are derived by combining the preceding ones in such a manner as to eliminate the quantity whose value is not sought.

TABLE.

No.	Given.	Required.	Formulas.
9.	n, r, s	a	$a = \frac{(r - 1)s}{r^n - 1}.$
10.	l, n, s	a	$a(s - a)^{n-1} = l(s - l)^{n-1}.$
11.	a, n, s	l	$l(s - l)^{n-1} = a(s - a)^{n-1}.$
12.	n, r, s	l	$l = \frac{(r - 1)sr^{n-1}}{r^n - 1}.$
13.	a, l, s	n	$n = \frac{\log l - \log a}{\log(s-a) - \log(s-l)} + 1$
14.	a, r, s	n	$n = \frac{\log[a + (r-1)s] - \log a}{\log r}.$
15.	l, r, s	n	$n = \frac{\log l - \log[lr - (r-1)s]}{\log r} + 1$
16.	a, n, s	r	$r^n - \frac{s}{a}r = 1 - \frac{s}{a}.$
17.	l, n, s	r	$r^n + \frac{s}{l - s}r^{n-1} = \frac{\quad}{l - s}.$
18.	a, n, r	s	$s = \frac{a(r^n - 1)}{r - 1}.$
19.	l, n, r	s	$s = \frac{lr^n - l}{r^n - r^{n-1}}.$
20.	a, l, n	s	$s = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$

☞ Of the twenty formulas in Geometrical Progression, the *first two* are *fundamental*, and should be *thoroughly committed* to memory; the *next six* are important in the solution of particular problems. The remainder are less practical.

407. By the fourth formula (Art. 404), any number of geometrical means may be found between two given quantities.

Let m = the number of means required.

Then $\qquad m+2=n.$

Substituting $m+2$ for n in the formula, we have

$$r=\left(\frac{l}{a}\right)^{\frac{1}{m+1}}.$$

The ratio being found, the means required are obtained by continued multiplication.

1. Find two geometrical means between 3 and 192.

SOLUTION. $r=\left(\frac{l}{a}\right)^{\frac{1}{m+1}}=\sqrt[3]{\frac{192}{3}}=\sqrt[3]{64}=4.$

The ratio being 4, the first mean is $3\times 4=12$; the second is

$$12\times 4=48.$$

2. Find three geometrical means between $\frac{1}{2}$ and 128.

PROBLEMS.

1. In a geometrical progression, the first term is 6, the last term 2916, and the ratio 3. What is the sum of all the terms?

2. In a decreasing geometrical series, the first term is $\frac{1}{2}$, the ratio $\frac{1}{3}$, and the number of terms 8. What is the sum of the series?

3. What is the sum of the series 1, 3, 9, 27, etc., to 15 terms?

4. Find the sum of 12 terms of the series, 1, $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$, etc.

5. If the first term of a series is 2, the ratio 3 and the number of terms 15, what is the last term?

6. What is the 16th term of a series, the first term of which is 3, and the ratio 3?

NOTE.—When the terms of the series are not stated directly, they may be represented algebraically.

7. Find three numbers in geometrical progression, such that their sum shall be 21, and the sum of their squares 189.

Let the three numbers be x, $\sqrt{xy}$, and y.

By the conditions,	$x+\sqrt{xy}+y = 21$	(1)
And	$x^2+xy+y^2 = 189$	(2)
Transposing and sq. (1),	$x^2+2xy+y^2 = 441-42\sqrt{xy}+xy$	(3)
Subtracting (2) from (3),	$xy = 252-42\sqrt{xy}+xy$	(4)
Transposing, etc.,	$\sqrt{xy} = 6$	(5)
Involving,	$xy = 36$	(6)
And	$3xy = 108$	(7)
Subtracting (7) from (2),	$x^2-2xy+y^2 = 81$	
Extracting root,	$x-y = 9$	(8)
Substituting (5) in (1),	$x+y = 15$	(9)
Combining (8) and (9),	$x = 12$	
	$y = 3$	

Hence the numbers, 12, 6, and 3, *Ans.*

8. A father gives his daughter $1 on New Year's day towards her portion, and doubles it on the first day of every month through the year. What is her portion?

9. A dairyman bought 10 cows, on the condition that he should pay 1 cent for the first, 3 for the second, 9 for the third, and so on to the last. What did he pay for the last cow and for the ten cows?

10. A man buys an umbrella, giving 1 cent for the first brace, 2 cents for the second brace, 4 for the third, and so on, there being 10 braces. What is the cost of the umbrella?

11. The sum of three numbers in geometrical progression is 26, and the sum of their squares 364. Find the numbers.

12. What would be the price of a horse, if he were to be sold for the 32 nails in his shoes, paying 1 mill for the first, 2 mills for the second, 4 for the third, and so on?

13. Find four numbers in geometrical progression, such that the sum of the first three is 130, and that of the last three is 390.

14. A man divides $210 in geometrical progression among three persons; the first had $90 more than the last. How much did each receive?

15. There are five numbers in geometrical progression. The sum of the first four is 468, and that of the last four is 2340. What are the numbers?

16. The sum of $700 is divided among 4 persons, whose shares are in geometrical progression; and the difference between the extremes is to the difference between the means as 37 to 12. What are the respective shares?

17. The population of a town increases annually in geometrical progression, rising in four years from 10000 to 14641. What is the ratio of annual increase?

18. The sum of four numbers in geometrical progression is 15, and the sum of their squares 85. What are the numbers?

HARMONICAL PROGRESSION.*

408. An ***Harmonical Progression*** is such, that of any three *consecutive* terms, the first is to the third as the difference of the first and second is to the difference of the second and third.

Thus, 10, 12, 15, 20, 30, 60,
are in harmonic progression; for

$$10 : 15 :: 12-10 : 15-12$$
$$12 : 20 :: 15-12 : 20-15$$
$$15 : 30 :: 20-15 : 30-20$$
$$20 : 60 :: 30-20 : 60-30$$

Let a, b, c, d, e, f, g, be an harmonical progression, then

$$a : c : a-b : b-c, \text{ etc.}$$

NOTE.—When three quantities are such, that the *first* is to the *third* as the *difference* of the first and second is to the *difference* of the second and third, they are said to be in *Harmonical Proportion.*

Thus, 2, 3, and 6, are in harmonical proportion.

408. What is an harmonical progression?

* If a musical string be divided in harmonical proportion, the different parts will vibrate in harmony. Hence, the name.

409. To Find the *Third Term* of an Harmonical Progression, the First Two being given.

Let a and b be the first two terms, and x the third term.

Then $a : x :: a-b : b-x$

Multiplying extremes, etc., $ab-ax = ax-bx$

Transposing, etc., $2ax-bx = ab$

Factoring, and dividing by $2a-b$, we have the

FORMULA. $$x = \frac{ab}{2a-b}.$$

RULE.—*Divide the product of the first two terms by twice the first minus the second term; the quotient will be the third term.*

NOTE.—This rule furnishes the means for extending an harmonic progression, by adding one term at a time to the two preceding terms.

1. Find the third term in the harmonic series of which 12 and 8 are the first two terms. *Ans.* 6.

2. Find the third term in the harmonic series of which 12 and 18 are the first two terms. *Ans.* 36.

3. If the first two terms of an harmonic progression are 15 and 20, what is the third term? *Ans.* 30.

4. Continue the series 12, 15, 20, for two terms. *Ans.* 30 and 60.

5. Continue the series $7\frac{1}{2}$, 9, 12, for two terms. *Ans.* 18 and 36.

410. To Find a *Mean* or Middle Term between Two Terms of an Harmonic Progression.

Let a and c be the first and third of three consecutive terms of an harmonic progression, and m the mean.

Then $a : c :: a-m : m-c$

Mult. extremes and means, $am-ac = ac-cm$

Transposing and uniting, $am+cm = 2ac$

Factoring and dividing by $a+c$, we have the

FORMULA. $$m = \frac{2ac}{a+c}.$$

---

409. How find the third term of an harmonical progression, the first two being given?

RULE.—*Divide twice the product of the first and third terms by their sum; the quotient will be the mean or middle term.*

6. The first and third of three consecutive terms of an harmonic progression are 9 and 18. Required the mean or middle term.

SOLUTION. $2\times9\times18=324$, and $9+18=27$.
Now $324\div27=12$, *Ans.*

7. Find an harmonic mean between 12 and 20. *Ans.* 15.
8. Find an harmonic mean between 15 and 30. *Ans.* 20.

411. The ***Reciprocals*** of the terms of an harmonic progression form an *arithmetical* progression.

Thus, the reciprocals of 10, 12, 15, 20, etc., viz.,

$$\tfrac{1}{10},\ \tfrac{1}{12},\ \tfrac{1}{15},\ \tfrac{1}{20},\ \tfrac{1}{30},\ \text{etc.},$$

are an arithmetical progression, whose common difference is $\frac{1}{60}$.

Again, let a, b, c be in harmonic progression.

Then $a : c :: a-b : b-c$

Mult. extremes and means, $ab-ac=ac-bc$

Dividing by abc, $\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}$. (Art. 383.)

Conversely, the reciprocals of an arithmetical progression form an harmonic progression. Thus,

The reciprocals of the arithmetical progression 1, 2, 3, 4, 5, etc., viz., $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc., are in harmonic progression.

412. If the *lengths* of *six musical strings* of *equal weight* and *tension*, are in the *ratio* of the numbers

$$1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ \tfrac{1}{6},\ \text{etc.},$$

the second will sound an *octave* above the first; the third will sound the *twelfth*; the fourth the *double octave*, etc.

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410. How find a mean between two terms of an harmonic progression?
411. What do the reciprocals of an harmonical progression form?

INFINITE SERIES.

413. An ***Infinite Series*** is one in which the *successive terms* are formed by some regular law, and the *number* of terms is *unlimited.*

414. A ***Converging Series*** is one the sum of whose terms, however great the number, cannot numerically *exceed* a *finite* quantity.

415. A ***Diverging Series*** is one the sum of whose terms is numerically *greater* than any *finite* quantity.

416. To Expand a Fraction into an *Infinite Series.*

REMARK.—Any common fraction whose exact value cannot be expressed by decimals, may be expanded into an infinite series.

1. Expand the fraction $\frac{1}{3}$ into an infinite series.

SOLUTION. $1 \div 3 = .333333$, and so on, to infinity.

Or, $1 \div 3 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000}$, etc. Hence, the

RULE.—*Divide the numerator by the denominator.*

2. Reduce $\frac{1}{1-x}$ to an infinite series.

$$\begin{array}{l}
1-x\,)\,1 \quad (\,1+x+x^2+x^3+x^4, \text{ etc., the quotient. (Art. 170.)} \\
\quad\quad \underline{1-x} \\
\quad\quad\quad +x \\
\quad\quad\quad \underline{+x-x^2} \\
\quad\quad\quad\quad +x^2 \\
\quad\quad\quad\quad \underline{+x^2-x^3} \\
\quad\quad\quad\quad\quad +x^3 \\
\quad\quad\quad\quad\quad \underline{+x^3-x^4} \\
\quad\quad\quad\quad\quad\quad +x^4, \text{ etc.}
\end{array}$$

Therefore, $\frac{1}{1-x} = 1+x+x^2+x^3+x^4+x^5$, etc., to infinity.

---

413. What is an infinite series? 414. A converging series? 415. Diverging? 416. How expand a fraction into an infinite series?

Let $x = \frac{1}{2}$; then will $\frac{1}{1-x} = \frac{1}{1-\frac{1}{2}} = 2$; and the series will be $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, etc., the sum of which $= 2$.

If $x = \frac{1}{3}$, then will $\frac{1}{1-x} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$, and the series will become

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}, \text{ etc.} = \frac{3}{2}.$$

NOTES.—1. If x is *less* than 1, the series will be *convergent*.

For, when x is *less* than 1, the remainder must continually decrease; therefore, the further the division is carried, the *less* will be the quantity to be added to the last term of the quotient in order to express the *exact value* of the fraction.

2. If x is *greater* than 1, the series will be *divergent*.

For, when x is greater than 1, the remainder must constantly increase; therefore, the *farther* the division is carried, the *greater* will be the quantity either positive or negative to be added to the quotient.

3. Reduce the fraction $\frac{1}{1+x}$ to an infinite series.

SOLUTION. $1 \div (1+x) = 1-x+x^2-x^3+x^4-x^5+$, etc.

☞ This series is the same as that in Ex. 2, except the odd powers of x are *negative*.

Let $x = \frac{1}{2}$; then will $\frac{1}{1+\frac{1}{2}} = \frac{2}{3}$; which is equal to the series

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+, \text{ etc.}$$

4. Reduce the fraction $\frac{1+x}{1-x}$ to an infinite series.

Ans. $1 + 2x + 2x^2 + 2x^3 + 2x^4$, etc.

417. A fraction whose denominator has more than two terms, may also be expanded into an infinite series.

5. Expand $\frac{1}{1-x+x^2}$ into an infinite series.

$$\begin{array}{l|l} 1-x+x^2\,) & 1 \qquad\qquad (1+x-x^3-x^4+x^6, \text{ etc., } Ans. \\ & \underline{1-x+x^2} \\ & \quad x-x^2 \\ & \quad \underline{x-x^2+x^3} \\ & \qquad\quad -x^3, \text{ etc.} \end{array}$$

418. To Expand a Compound Surd into an *Infinite Series.*

6. Reduce $\sqrt{1+x}$ to an infinite series.

OPERATION.

$$
\begin{array}{r|l}
 & 1 + x \ \big|\ 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}, \text{ etc., } \textit{Ans.} \\
 & 1 \\
2 + \frac{x}{2} & +x \\
 & +x + \frac{x^2}{4} \\
2 + x - \frac{x^2}{8} & -\frac{x^2}{4} \\
 & -\frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{64} \\
2 + x - \frac{x^2}{4} + \frac{x^3}{16} & +\frac{x^3}{8} - \frac{x^4}{64}, \text{ etc.}
\end{array}
$$

Hence, the

RULE.—*Extract the square root of the given surd.* (Art. 298.)

7. Expand $\sqrt{x^2-y^2}$. *Ans.* $x - \frac{y^2}{2x} - \frac{y^4}{8x^3} - \frac{y^6}{16x^5}$, etc.

8. Expand $\sqrt{2}$, or $\sqrt{1+1}$. *Ans.* $1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16}$, etc.

419. The Binomial Theorem applied to the Formation of Infinite Series.

The Binomial Theorem may often be employed with advantage, in finding the roots of binomials. For a root is expressed like a power, except the exponent of one is an *integer*, and that of the other is a *fraction.*

9. Expand $(x+y)^{\frac{1}{2}}$ into an infinite series.

SOLUTION.—The terms without coefficients are

$$x^{\frac{1}{2}},\quad x^{-\frac{1}{2}}y,\quad x^{-\frac{3}{2}}y^2,\quad x^{-\frac{5}{2}}y^3,\quad x^{-\frac{7}{2}}y^4,\quad \text{etc.}$$

The coefficient of the second term is $+\frac{1}{2}$; of the 3d, $\frac{\frac{1}{2} \times -\frac{1}{2}}{2} = -\frac{1}{8}$; of the 4th term, $\frac{-\frac{1}{8} \times -\frac{3}{2}}{3} = +\frac{1}{16}$, etc.

The series is $x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}y - \frac{1}{8}x^{-\frac{3}{2}}y^2 + \frac{1}{16}x^{-\frac{5}{2}}y^3$, etc.

---

418. How expand a surd into an infinite series?

420. When the *index* of the required power of a binomial is a *positive integer*, the series will *terminate*. For, the index of the leading quantity continually decreases by 1; and soon becomes 0; then the series must stop. (Art. 269.)

421. When the index of the required power is *negative*, the series will never terminate. For, by the successive subtractions of a *unit* from the index, it will never become 0; and the series may be continued indefinitely.

10. Expand $(x^2+y)^{\frac{1}{2}}$ into an infinite series, keeping the factors of the coefficients distinct.

$$(x^2+y)^{\frac{1}{2}} = x + \frac{y}{2x} - \frac{y^2}{2.4x^3} + \frac{3y^3}{2.4.6x^5} - \frac{3.5y^4}{2.4.6.8x^7}, \text{ etc.}$$

11. Expand $\sqrt{2}$, or $(1+1)^{\frac{1}{2}}$, keeping the factors of the coefficients distinct.

Ans. $1+\frac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}+\frac{3.5.7}{2.4.6.8.10}$, etc.

422. An *Infinite Series* must not be confounded with an *Infinite Quantity*.

423. An ***Infinite Quantity*** is a quantity so great that nothing can be *added* to it.

424. An ***Infinite Series*** is a series in which the *number of terms* is *unlimited*.

425. The *magnitude* of the former admits of *no increase*; while in the latter the *number of terms* admits of *no increase*, and yet the sum of all the terms may be a small quantity.

Thus, if the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, etc., in which each succeeding term is *half* the preceding, is continued to *infinity*, the *sum* of all the terms cannot exceed a *unit*.

426. When one quantity continually approximates another without reaching it, the latter is called the ***Limit*** of the former.

427. An ***Infinitesimal*** is a quantity whose value is *less* than any *assignable quantity*.

428. The ***Sign of Infinity,*** or of an infinite quantity, is a character resembling an horizontal figure eight (∞).

The ***Sign of an Infinitesimal*** is zero (0).

429. One infinite series may be *greater* or *less* than another.

Thus, the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, etc., whose limit is 2, is greater than the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$, etc., whose limit is 1.

430. Since an *infinitesimal* is less than *any assignable quantity*, and next to nothing, when connected with finite quantities by the sign $+$ or $-$, it is of no account, and may be rejected.

431. An infinite series may be *multiplied* by a finite quantity.

Thus, if the series 222222, etc., is multiplied by 3,
the product 666666, etc., is three times the multiplicand.

432. An infinite series may also be *divided* by a finite quantity.

Thus, if the series 888888, etc., is divided by 2,
the quotient 444444, etc., is half the dividend.

433. If a *finite* quantity is multiplied by an *infinitesimal*, the *product* will be an *infinitesimal*. For, with a given multiplicand, the *less* the multiplier, the *less* will be the product. Thus, $x\times 0 = 0$.

434. If a *finite* quantity is *divided* by an *infinitesimal*, the *quotient* will be *infinite*. Thus, $x \div 0 = \infty$.

If a *finite* quantity is divided by an *infinite* quantity, the quotient will be an *infinitesimal*. Thus, $x \div \infty = 0$.

If an *infinitesimal* is divided by a *finite* quantity, the quotient is an *infinitesimal*. Thus, $0 \div x = 0$.

NOTE.—In higher mathematics, the expression $0 \div 0$ admits of various interpretations.

435. To Find the *Sum* of a Converging Infinite Series, the First Term and Ratio being given.

By the second formula in geometrical progression, we have for an *increasing* series (Art. 402),

$$s = \frac{lr - a}{r - 1}, \quad \text{or} \quad \frac{ar^n - a}{r - 1}.$$

In a *decreasing* series, the ratio r is less than 1; therefore, l or ar^{n-1}, is less than a. (Art. 398.)

That both terms of the fraction $\frac{lr-a}{r-1}$, or $\frac{ar^n-a}{r-1}$ may be *positive*, we change the signs of both (Art. 166), and

$$s = \frac{a - lr}{1 - r}.$$

But, in a decreasing infinite series, l becomes an infinitesimal, or 0; therefore, $lr = 0$. (Art. 427.) Hence, rejecting the infinitesimal from $s = \frac{a - lr}{1 - r}$, we have the

FORMULA. $$s = \frac{a}{1 - r}.$$

RULE.—*Divide the first term by* 1 *minus the ratio.*

1. Find the sum of the infinite series

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81}, \text{ etc.}$$

2. Required the sum of the infinite series

$$1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} +, \text{ etc.}$$

3. Find the sum of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$, etc.
4. Find the sum of the infinite series $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} +$, etc.
5. Find the sum of the series $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} +$, etc.
6. Find the sum of the series $3 + 2 + \frac{4}{3} +$, etc.
7. Find the sum of the series $4 + \frac{12}{5} + \frac{36}{25} +$, etc.
8. Find the sum of the series .3333, etc.
9. Find the sum of the series .66666, etc.
10. Find the sum of the series $\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} +$ etc.

11. Suppose a ball to be put in motion by a force which impels it 10 rods the first second, 8 rods the next, and so on, decreasing by a ratio of $\frac{4}{5}$ each second to infinity. Through what space would it move?

CHAPTER XIX.

LOGARITHMS.*

436. The ***Logarithm*** of a number is the *exponent* of the power to which a given *fixed* number must be raised to produce that number.

437. This *Fixed Number* is called the ***Base*** of the system.

Thus, if 3 is the base, then 2 is the logarithm of 9, because $3^2 = 9$; and 3 is the logarithm of 27, because $3^3 = 27$, and so on.

Again, if 4 is the base, then 2 is the logarithm of 16, because $4^2 = 16$; and 3 is the logarithm of 64, because $4^3 = 64$, and so on.

438. In forming a *system* of logarithms, any number, except 1, may be taken as the base, and when the base is selected, all other numbers are considered as some *power* or *root* of this base. Hence, there may be an unlimited number of systems.

NOTE.—Since all *powers* and *roots* of 1 are 1, it is obvious that other numbers cannot be represented by its powers or roots. (Art. 289.)

439. There are two systems of logarithms in use, the Napierian system,† the base of which is 2.718281828, and the Common System, whose base is 10.‡

☞ The abbreviation *log.* stands for the term logarithm.

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436. What are logarithms? 437. What is this fixed number called? 439. Name the systems in use. The base of each.

* The term *logarithm* is derived from two Greek words, meaning the *relation* of *numbers*.

† So called from Baron Napier, of Scotland, who invented logarithms in 1614.

‡ The common system was invented by Henry Briggs, an English mathematician, in 1624.

440. The ***Base*** of common logarithms being 10, all other numbers are considered as powers or roots of 10.

Thus, the log. of 1 is 0; for 10^0 equals 1 (Art. 259);
" " 10 is 1; for 10^1 " 10;
" " 100 is 2; for 10^2 " 100;
" " 1000 is 3; for 10^3 " 1000, etc. Hence,

The logarithm of any number between 1 and 10 is a fraction; for any number between 10 and 100, the logarithm is 1 plus a fraction; and for any number between 100 and 1000, the logarithm is 2 plus a fraction, and so on.

441. By means of negative exponents, this principle may be applied to fractions.

Thus (Art. 256), the log. of .1 is -1; for 10^{-1} equals .1;
" " .01 is -2; for 10^{-2} " .01;
" " .001 is -3; for 10^{-3} " .001.

Therefore, the logarithms for all numbers between 1 and 0.1 lie between 0 and -1, and are respectively equal to -1 plus a fraction; for any number between 0.1 and 0.01, the logarithm is -2 plus a fraction; and for any number between 0.01 and 0.001, the logarithm is -3 plus a fraction, and so on.

Hence, the logarithms of all numbers greater than 10 or less than 1, and not exact powers of 10, are composed of two parts, an *integer* and a *fraction.*

Thus, the logarithm of 28 is 1.44716;
and of .28 is $\bar{1}.44716$.

442. The *integral* part of a logarithm is called the ***Characteristic;*** the *decimal* part, the ***Mantissa.***

443. The ***Characteristic*** of the logarithm of a whole number is one less than the number of integral figures in the given number.

Thus, the characteristic of the logarithm of 49 is 1; that of 495 is 2; that of 4956 is 3; that of 6256414 is also 3, etc.

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440. What is the logarithm of any number from 1 to 10? From 10 to 100? From 100 to 1000? 442. What is the integral part of a logarithm called? The decimal part? 443. What is the characteristic of the logarithm of a whole number?

444. The ***Characteristic*** of the logarithm of a *decimal* is *negative*, and is one *greater* than the number of ciphers before the first significant figure of the fraction.

Thus, the characteristic of the logarithm of $\frac{1}{10}$ or .1 is -1; that of $\frac{1}{100}$, or .01, is -2; that of $\frac{1}{1000}$, or .001, is -3, etc. (Art. 256.)

The logarithm of .2 is -1 with a decimal added to it; that of .05 is -2 with a decimal added to it, etc.

NOTE.—It should be observed that the *characteristic only* is *negative*, while the mantissa, or *decimal* part, is always *positive*. To indicate this, the sign $-$ is placed *over* the characteristic, instead of *before* it.

Thus, the logarithm of .2 is $\bar{1}.30103$,
" " " .05 is $\bar{2}.69897$, etc.

445. The ***Decimal Part*** of the logarithm of any number is the same as the logarithm of the number multiplied or divided by 10, 100, 1000, etc.

Thus, the logarithm of 1876 is 3.27325; of 18760 is 4.27325, etc.

TABLES OF LOGARITHMS.

446. A ***Table of Logarithms*** is one which contains the logarithms of all numbers between given limits.

447. The Table found on the following pages gives the mantissas of common logarithms to five decimal places for all numbers from 1 to 1000, inclusive.

The *characteristics* are omitted, and must be supplied by inspection. (Arts. 443, 444.)

NOTES.—1. The first decimal figure in column 0 is often the same for several successive numbers, but is printed only once, and is understood to belong to each of the blank places below it.

2. The character (•) shows that the figure belonging to the place it occupies has changed from 9 to 0, and through the rest of this line the first figure of the mantissa stands in the next line below.

444. What is the characteristic of the logarithm of a decimal? 445. What is the effect upon the decimal part of the log. of a number, if the number is multiplied or divided by 10, 100, 1000, etc. 446. What is a table of logarithms?

448. To Find the *Logarithm* of any Number from 1 to 10.

RULE.—*Look for the given number in the first line of the table; its logarithm will be found directly below it.*

1. Find the logarithm of 7. *Ans.* 0.84510.
2. Find the logarithm of 9. *Ans.* 0.95424.

449. To Find the *Logarithm* of any Number from 10 to 1000, inclusive.

RULE.—*Look in the column marked N for the first two figures of the given number, and for the third at the head of one of the other columns.*

Under this third fiqure, and opposite the first two, will be found the last decimal figures of the logarithm. The first one is found in the column marked 0.

To this decimal prefix the proper characteristic. (Art. 443.)

NOTE.—If the number contains 4 or more figures, multiply the tabular difference by the remaining figures, and rejecting from the right of the product as many figures as you multiply by, add the rest to the log. of the first 3 figures.

3. Find the logarithm of 108. *Ans.* 2.03342.
4. Find the logarithm of 176. *Ans.* 2.24551.
5. Find the logarithm of 1999. *Ans.* 3.30085.

450. To Find the Logarithm of a *Decimal Fraction.*

RULE.—*Take out the logarithm of a whole number consisting of the same figures, and prefix to it the proper negative characteristic.* (Art. 444.)

NOTE.—If the number consists of an *integer* and a *decimal*, find the logarithm in the same manner as if *all* the figures were integers, and prefix the characteristic which belongs to the *integral* part. (Art. 443.)

6. What is the log. of 0.95? *Ans.* $\bar{1}.97772$.
7. What is the log. of 0.0125? *Ans.* $\bar{2}.09691$.
8. What is the log. of 0.0075? *Ans.* $\bar{3}.87506$.
9. What is the log. of 16.45? *Ans.* 1.21616.
10. What is the log. of 185.3? *Ans.* 2.26787.

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448. How find the logarithm of a number from 1 to 10? 449. From 10 to 1000? 450. How find the log. of a decimal? *Note.* Of an integer and a decimal?

451. To Find the *Number* belonging to a given Logarithm.

RULE.—*Look for the decimal figures of the given logarithm in the table under the column marked* 0; *and if all of them are not found in that column, look in the other columns on the right till you find them exactly, or very nearly; directly opposite, in the column marked N, will be found the first two figures, and at the top, over the logarithm, the third figure of the given number.*

Make this number correspond to the characteristic of the given logarithm, by pointing off decimals, or by adding ciphers, if necessary, and it will be the number required.

NOTE.—If the characteristic of a logarithm is *negative*, the number belonging to it is a *fraction*, and as many ciphers must be prefixed to the number found in the table, as there are *units* in the characteristic *less* 1. (Art. 444.)

452. When the *Decimal Part* of the given Logarithm is not exactly, or very nearly, found in the Table.

RULE.—*From the given logarithm subtract the next less logarithm found in the tables; annex ciphers to the remainder, and divide it by the tabular difference* (*marked D*) *as far as necessary.*

To the number belonging to the less logarithm annex the quotient, and make the number thus produced correspond to the characteristic of the given logarithm, as above.

NOTE.—For every cipher annexed to the remainder, either a *significant* figure or a *cipher* must be put in the quotient.

11. What number belongs to 2.17231? *Ans.* 148.7.
12. What number belongs to 1.25261? *Ans.* 17.89.
13. What number belongs to 3.27715? *Ans.* 1893.
14. What number belongs to 2.30963? *Ans.* 204.
15. What number belongs to 4.29797? *Ans.* 19858.29.
16. What number belongs to $\bar{1}.14488$? *Ans.* 0.1396.
17. What number belongs to $\bar{2}.29136$? *Ans.* 0.01956.
18. What number belongs to $\bar{3}.30928$? *Ans.* 0.002038.

451. How find the number belonging to a logarithm?

453. Computations by logarithms are based upon the following principles:

1°. *The sum of the logarithms of two numbers is equal to the logarithm of their product.*

Let a and c denote any two numbers, m and n their logarithms, and b the base.

Then	$b^m = a$
And	$b^n = c$
Multiplying,	$b^{m+n} = ac.$

2°. *The logarithm of the dividend diminished by the logarithm of the divisor is equal to the logarithm of the quotient of the two numbers.*

Let a and c denote any two numbers, m and n their logarithms, and b the base.

Then	$b^m = a$
And	$b^n = c$
Dividing,	$b^{m-n} = a \div c.$

454. To *Multiply* by Logarithms.

1. Required the product of 35 by 23.

SOLUTION.—The log. of 35 = 1.54407
" " " 23 = 1.36173

Adding, 2.90580. (Art. 453, Prin. 1.)

The number belonging is 805, *Ans.* Hence, the

RULE—*Add the logarithms of the factors; the sum will be the logarithm of the product.*

NOTES.—1. If the sum of the decimal parts exceeds 9, add the *tens* figure to the characteristic.

2. If either or all the characteristics are *negative*, they must be added according to Art. 65. But as the mantissa is always *positive*, that which is carried from the mantissa to the characteristic must be considered *positive*.

2. What is the product of 109.3 by 14.17?
3. What is the product of 1.465 by 1.347?
4. What is the product of .074 by 1500?

453. Upon what two principles are computations by logarithms based? 454. How multiply by logarithms?

455. To *Divide* by Logarithms.

5. Required the quotient of 120 by 15.

SOLUTION.—The log. of 120 = 2.07918
" " " 15 = 1.17609
" " " quotient = 0.90309. *Ans.* 8. Hence, the

RULE.—*From the logarithm of the dividend subtract the logarithm of the divisor; the difference will be the logarithm of the quotient.* (Art. 453, Prin. 2.)

NOTES.—1. When either of the characteristics is *negative*, or when the lower one is greater than the one above it, change the sign of the subtrahend, and proceed as in addition.

2. When 1 is carried from the mantissa to the characteristic, it must be considered *positive*, and be added to the characteristic *before* the sign is changed.

6. What is the quotient of 12.48 by 0.16?
7. What is the quotient of .045 by 1.20?
8. What is the quotient of 1.381 by .096?

456. *Negative* quantities are divided in the same manner as *positive* quantities.

If the sign of the divisor is the same as that of the dividend, prefix the sign + to the quotient; but if different, prefix the sign —.

9. Divide —128 by —47.
10. Divide —186 by —0.064.
11. Divide —0.156 by —0.86.
12. Divide —0.194 by 0.042.

457. To *Involve* a Number by Logarithms.

Multiplication by logarithms is performed by *addition.* (Art. 453.) Therefore, if the logarithm of a quantity is added *to itself once*, the result will be the logarithm of the *second power* of that quantity; if added to itself *twice*, the result will be the *third* power of that quantity, and so on. Hence,

RULE.—*Multiply the logarithm of the number by the exponent of the required power.*

455. How divide by them? 457. How involve a number by logarithms?

NOTES.—1. This rule depends upon the principle that logarithms are the *exponents* of *powers* and *roots*, and a power or root is involved by multiplying its *index* into the *index* of the power required.

2. In this rule, whatever is carried from the mantissa to the characteristic is *positive*, whether the index itself is *positive* or *negative*.

13. What is the cube of 1.246.

SOLUTION.—The log. of 1.246 is	0.09551	
Index of the required power is	3	
Log. of power is	0.28653.	*Ans.* 1.93435.

14. What is the fourth power of .135?

15. What is the tenth power of 1.42?

16. What is the twenty-fifth power of 1.234?

458. To *Extract the Root* of a Number by Logarithms.

A quantity is resolved into any number of *equal factors*, by dividing its index into as many *equal parts*. (Art. 281.) Hence, the

RULE.—*Divide the logarithm of the number by the index of the required root.*

NOTE.—This rule depends upon the principle that the *root* of a quantity is found by *dividing the exponent* by the number expressing the required root.

17. What is the square root of 1.69?

SOLUTION.—The log. of 1.69 is	0.22789	
The index is 2,	2) .22789	
Logarithm of root,	0.11394.	*Ans.* 1.3.

18. What is the cube root of 143.2?

19. What is the sixth root of 1.62?

20. What is the eighth root of 1549?

21. What is the tenth root of 1876?

459. If the characteristic of the logarithm is *negative*, and cannot be *divided* by the *index* of the required root without a *remainder*, make it *positive* by adding to the *characteristic* such a *negative* number as will make it *exactly divisible* by the *divisor*, and *prefix* an equal *positive* number to the decimal part of the logarithm.

---

458. How extract the root?

22. It is required to find the cube root of .0164.

SOLUTION.—The log. of .0164 is $\bar{2}.21484$.

Preparing the log., $3\,)\,\bar{3}+1.21484$

$\bar{1}.40494$. *Ans.* 0.25406+.

23. What is the sixth root of .001624?

24. What is the seventh root of .01449?

25. What is the eighth root of .0001236?

460. To Calculate *Compound Interest* by Logarithms.

RULE.—*Find the amount of* 1 *dollar for* 1 *year; multiply its logarithm by the number of years, and to the product add the logarithm of the principal. The sum will be the logarithm of the amount for the given time.*

From the amount subtract the principal, and the remainder will be the interest.

NOTES.—1. If the interest becomes due *half yearly* or *quarterly*, find the amount of one dollar for the half year or quarter, and multiply the logarithm by the number of half years or quarters in the time.

2. This rule is based upon the principle that the several *amounts* in compound interest form a *geometrical series*, of which the principal is the first term, the amount of \$1 for 1 year the ratio, and the number of years + 1 the number of terms

26. What is the amount of \$1565 for 40 years, at 6 per cent compound interest?

SOLUTION.—The amt. of \$1 for 1 year is \$1.06; its log.,	0.02531
The number of years,	40
Product,	1.01240
The given principal, \$1565; its log.,	3.19453
Ans. \$16103.78.	4.20693

27. What is the amount of \$1500, at 7 per cent compound interest, for 4 years? *Ans.* \$1966.05.

28. What is the amount of \$370, at 5 per cent compound interest for 33 years? *Ans.* \$1851.27+.

---

460. How calculate compound interest by logarithms?

TABLE OF COMMON LOGARITHMS.

N.	0	1	2	3	4	5	6	7	8	9	D.
	.00000	.00000	.30103	.47712	.60206	.69897	.77815	.84510	.90309	.95424	
10	.00000	0432	0860	1284	1703	2119	2531	2938	3342	3743	416
11	4139	4532	4922	5308	5691	6070	6446	6819	7188	7555	379
12	7918	8279	8636	8991	9342	9691	•037	•380	•721	1059	349
13	.11394	1727	2057	2385	2711	3033	3354	3672	3988	4302	322
14	4613	4922	5229	5534	5836	6137	6435	6732	7026	7319	301
15	7609	7898	8184	8469	8752	9033	9313	9590	9866	•140	281
16	.20412	0683	0952	1219	1484	1748	2011	2272	2531	2789	264
17	3045	3300	3553	3805	4055	4304	4551	4797	5042	5285	249
18	5527	5768	6007	6245	6482	6717	6951	7184	7416	7646	235
19	7875	8103	8330	8556	8780	9003	9226	9447	9667	9885	223
20	.30103	0320	0535	0750	0963	1175	1387	1597	1806	2015	212
21	2222	2428	2634	2838	3041	3244	3445	3646	3846	4044	203
22	4242	4439	4635	4831	5025	5218	5411	5603	5794	5984	193
23	6173	6361	6549	6736	6922	7107	7291	7475	7658	7840	185
24	8021	8202	8382	8561	8739	8917	9094	9270	9445	9620	178
25	9794	9967	•140	•312	•483	•654	•824	•993	1162	1330	171
26	.41497	1664	1830	1996	2160	2325	2488	2651	2814	2975	165
27	3136	3297	3457	3616	3775	3933	4091	4248	4405	4560	158
28	4716	4871	5025	5179	5332	5485	5637	5788	5939	6090	153
29	6240	6389	6538	6687	6835	6982	7129	7276	7422	7567	147
30	.47712	7857	8001	8144	8287	8430	8572	8714	8855	8996	143
31	9136	9276	9416	9554	9693	9831	9969	•106	•243	•379	138
32	.50515	0651	0786	0920	1055	1188	1322	1455	1587	1720	133
33	1851	1983	2114	2244	2375	2505	2634	2763	2892	3020	130
34	3148	3275	3403	3529	3656	3782	3908	4033	4158	4283	126
35	4407	4531	4654	4778	4900	5023	5145	5267	5388	5509	123
36	5630	5751	5871	5991	6110	6229	6348	6467	6585	6703	119
37	6820	6937	7054	7171	7287	7403	7519	7634	7749	7864	116
38	7978	8093	8206	8320	8433	8546	8659	8771	8883	8995	113
39	9107	9218	9329	9439	9550	9660	9770	9879	9988	•097	110
40	.60206	0314	0423	0531	0638	0746	0853	0959	1066	1172	108
41	1278	1384	1490	1595	1700	1805	1909	2014	2118	2221	105
42	2325	2428	2531	2634	2737	2839	2941	3043	3144	3246	102
43	3347	3448	3548	3649	3749	3849	3949	4048	4147	4247	100
44	4345	4444	4542	4640	4738	4836	4934	5031	5128	5225	98
45	5321	5418	5514	5610	5706	5801	5897	5992	6087	6181	95
46	6276	6370	6464	6558	6652	6745	6839	6932	7025	7117	93
47	7210	7302	7394	7486	7578	7669	7761	7852	7943	8034	91
48	8124	8215	8305	8395	8485	8574	8664	8753	8842	8931	89
49	9020	9108	9197	9285	9373	9461	9548	9636	9723	9810	88
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
50	.69897	9984	◆070	◆157	◆243	◆329	◆415	◆501	◆586	◆672	86
51	.70757	0842	0927	1012	1096	1181	1265	1349	1433	1517	85
52	1600	1684	1767	1850	1933	2016	2099	2181	2263	2346	83
53	2428	2510	2591	2673	2754	2835	2917	2997	3078	3159	81
54	3239	3320	3400	3480	3560	3640	3719	3799	3878	3957	80
55	.74036	4115	4194	4273	4351	4429	4508	4586	4663	4741	78
56	4819	4896	4974	5051	5128	5205	5282	5358	5435	5511	77
57	5588	5664	5740	5815	5891	5967	6042	6118	6193	6268	76
58	6343	6418	6492	6567	6641	6716	6790	6864	6938	7012	75
59	7085	7159	7232	7306	7379	7452	7525	7597	7670	7743	73
60	.77815	7887	7960	8032	8104	8176	8247	8319	8390	8462	72
61	8533	8604	8675	8746	8817	8888	8958	9029	9099	9169	71
62	9239	9309	9379	9449	9519	9588	9657	9727	9796	9865	69
63	9934	◆003	◆072	◆140	◆209	◆277	◆346	◆414	◆482	◆550	68
64	.80618	0686	0754	0821	0889	0956	1023	1090	1158	1225	67
65	1291	1358	1425	1491	1558	1624	1690	1757	1823	1889	66
66	1954	2020	2086	2151	2217	2282	2347	2413	2478	2543	65
67	2608	2672	2737	2802	2866	2930	2995	3059	3123	3187	64
68	3251	3315	3378	3442	3506	3569	3632	3696	3759	3822	63
69	3885	3948	4011	4073	4136	4199	4261	4323	4386	4448	63
70	.84510	4572	4634	4696	4757	4819	4881	4942	5003	5065	62
71	5126	5187	5248	5309	5370	5431	5491	5552	5612	5673	61
72	5733	5794	5854	5914	5974	6034	6094	6153	6213	6273	60
73	6332	6392	6451	6510	6570	6629	6688	6747	6806	6864	59
74	6923	6982	7040	7099	7157	7216	7274	7332	7390	7448	59
75	7506	7564	7622	7680	7737	7795	7852	7910	7967	8024	58
76	8081	8139	8196	8253	8309	8366	8423	8480	8536	8593	57
77	8649	8705	8762	8818	8874	8930	8986	9042	9098	9154	56
78	9210	9265	9321	9376	9432	9487	9542	9598	9653	9708	55
79	9763	9818	9873	9927	9982	◆037	◆091	◆146	◆200	◆255	55
80	.90309	0363	0417	0472	0526	0580	0634	0687	0741	0795	54
81	0849	0902	0956	1009	1062	1116	1169	1222	1275	1328	54
82	1381	1434	1487	1540	1593	1645	1698	1751	1803	1856	52
83	1908	1960	2012	2065	2117	2169	2221	2273	2324	2376	52
84	2428	2480	2531	2583	2634	2686	2737	2788	2840	2891	52
85	2942	2993	3044	3095	3146	3197	3247	3298	3349	3399	51
86	3450	3500	3551	3601	3651	3702	3752	3802	3852	3902	51
87	3952	4002	4052	4101	4151	4201	4250	4300	4350	4399	50
88	4448	4498	4547	4596	4645	4694	4743	4792	4841	4890	49
89	4939	4988	5037	5085	5134	5182	5231	5279	5328	5376	48
90	.95424	5473	5521	5569	5617	5665	5713	5761	5809	5856	48
91	5904	5952	6000	6047	6095	6142	6190	6237	6284	6332	47
92	6379	6426	6473	6520	6567	6614	6661	6708	6755	6802	47
93	6848	6895	6942	6988	7035	7081	7128	7174	7220	7267	46
94	7313	7359	7405	7451	7497	7543	7589	7635	7681	7727	46
95	7772	7818	7864	7909	7955	8000	8046	8091	8137	8182	45
96	8227	8272	8318	8363	8408	8453	8498	8543	8588	8632	45
97	8677	8722	8767	8811	8856	8901	8945	8990	9034	9078	45
98	9123	9167	9211	9255	9300	9344	9388	9432	9476	9520	44
99	9564	9607	9651	9695	9739	9782	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.

CHAPTER XX.

MATHEMATICAL INDUCTION AND BUSINESS FORMULAS.

461. ***Mathematical Induction*** consists in proving by trial that a proposition is true in a certain case; and, finding it true in the next case, then in the third, and so on, we conclude it must be *true* in *all similar* cases.

462. Many of the principles and formulas of Arithmetic and Algebra are established by this mode of reasoning.

463. Take the familiar principle in Arithmetic:

The product of any two or more numbers is the same, in whatever order the factors are taken.

To prove this principle of two numbers, as 5 and 3, the pupil represents the number 5 by as many unit marks in a horizontal row, and under this places two similar rows.

$$\begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}$$

He sees that the *number* of marks in the *horizontal row* taken 3 times is equal to the number of marks in a *perpendicular* row taken 5 times; that is, 3 times 5 = 5 times 3.

He then takes *three* factors and finds the proposition true, and so on. Hence, he concludes the principle is *universally true.*

464. Next, suppose it be asserted:

The product of the sum and difference of two quantities is equal to the difference of their squares. (Art. 103.)

Taking two quantities, as $4+3$ and $4-3$, or $a+b$ and $a-b$.

$$\begin{array}{ll} \text{Multiply} & 4+3 \\ \text{By} & \underline{4-3\qquad\quad} \\ & 4^2+4\times 3 \\ & \underline{\qquad -4\times 3-3^2} \\ \text{Product,} & 4^2 \qquad\qquad -3^2 \end{array} \qquad\qquad \begin{array}{ll} \text{Or} & a+b \\ \text{By} & \underline{a-b\qquad} \\ & a^2+ab \\ & \underline{\qquad -ab-b^2} \\ & a^2 \qquad\quad -b^2 \end{array}$$

461. In what does Mathematical induction consist? Illustration.

N.	0	1	2	3	4	5	6	7	8	9	D.
50	.69897	9984	•070	•157	•243	•329	•415	•501	•586	•672	86
51	.70757	0842	0927	1012	1096	1181	1265	1349	1433	1517	85
52	1600	1684	1767	1850	1933	2016	2099	2181	2263	2346	83
53	2428	2510	2591	2673	2754	2835	2917	2997	3078	3159	81
54	3239	3320	3400	3480	3560	3640	3719	3799	3878	3957	80
55	.74036	4115	4194	4273	4351	4429	4508	4586	4663	4741	78
56	4819	4896	4974	5051	5128	5205	5282	5358	5435	5511	77
57	5588	5664	5740	5815	5891	5967	6042	6118	6193	6268	76
58	6343	6418	6492	6567	6641	6716	6790	6864	6938	7012	75
59	7085	7159	7232	7306	7379	7452	7525	7597	7670	7743	73
60	.77815	7887	7960	8032	8104	8176	8247	8319	8390	8462	72
61	8533	8604	8675	8746	8817	8888	8958	9029	9099	9169	71
62	9239	9309	9379	9449	9519	9588	9657	9727	9796	9865	69
63	9934	•003	•072	•140	•209	•277	•346	•414	•482	•550	68
64	.80618	0686	0754	0821	0889	0956	1023	1090	1158	1225	67
65	1291	1358	1425	1491	1558	1624	1690	1757	1823	1889	66
66	1954	2020	2086	2151	2217	2282	2347	2413	2478	2543	65
67	2608	2672	2737	2802	2866	2930	2995	3059	3123	3187	64
68	3251	3315	3378	3442	3506	3569	3632	3696	3759	3822	63
69	3885	3948	4011	4073	4136	4199	4261	4323	4386	4448	63
70	.84510	4572	4634	4696	4757	4819	4881	4942	5003	5065	62
71	5126	5187	5248	5309	5370	5431	5491	5552	5612	5673	61
72	5733	5794	5854	5914	5974	6034	6094	6153	6213	6273	60
73	6332	6392	6451	6510	6570	6629	6688	6747	6806	6864	59
74	6923	6982	7040	7099	7157	7216	7274	7332	7390	7448	59
75	7506	7564	7622	7680	7737	7795	7852	7910	7967	8024	58
76	8081	8139	8196	8253	8309	8366	8423	8480	8536	8593	57
77	8649	8705	8762	8818	8874	8930	8986	9042	9098	9154	56
78	9210	9265	9321	9376	9432	9487	9542	9598	9653	9708	55
79	9763	9818	9873	9927	9982	•037	•091	•146	•200	•255	55
80	.90309	0363	0417	0472	0526	0580	0634	0687	0741	0795	54
81	0849	0902	0956	1009	1062	1116	1169	1222	1275	1328	54
82	1381	1434	1487	1540	1593	1645	1698	1751	1803	1856	52
83	1908	1960	2012	2065	2117	2169	2221	2273	2324	2376	52
84	2428	2480	2531	2583	2634	2686	2737	2788	2840	2891	52
85	2942	2993	3044	3095	3146	3197	3247	3298	3349	3399	51
86	3450	3500	3551	3601	3651	3702	3752	3802	3852	3902	51
87	3952	4002	4052	4101	4151	4201	4250	4300	4350	4399	50
88	4448	4498	4547	4596	4645	4694	4743	4792	4841	4890	49
89	4939	4988	5037	5085	5134	5182	5231	5279	5328	5376	48
90	.95424	5473	5521	5569	5617	5665	5713	5761	5809	5856	48
91	5904	5952	6000	6047	6095	6142	6190	6237	6284	6332	47
92	6379	6426	6473	6520	6567	6614	6661	6708	6755	6802	47
93	6848	6895	6942	6988	7035	7081	7128	7174	7220	7267	46
94	7313	7359	7405	7451	7497	7543	7589	7635	7681	7727	46
95	7772	7818	7864	7909	7955	8000	8046	8091	8137	8182	45
96	8227	8272	8318	8363	8408	8453	8498	8543	8588	8632	45
97	8677	8722	8767	8811	8856	8901	8945	8990	9034	9078	45
98	9123	9167	9211	9255	9300	9344	9388	9432	9476	9520	44
99	9564	9607	9651	9695	9739	9782	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6				

CHAPTER XX.

MATHEMATICAL INDUCTION AND BUSINESS FORMULAS.

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* * * * *
* * * * *
* * * * *

He sees that the *number* of marks in the *horizontal row* taken 3 times is equal to the number of marks in a *perpendicular* row taken 5 times; that is, 3 times 5 = 5 times 3.

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Taking two quantities, as $4+3$ and $4-3$, or $a+b$ and $a-b$.

Multiply	$4 + 3$	Or	$a + b$
By		By	$a - b$
			$a^2 + ab$
	$- 3^2$		$- ab - b^2$
	$- 3^2$		a^2 [illegible]

[illegible] tical induction consist? Illustrati[illegible]

444. The ***Characteristic*** of the logarithm of a *decimal* is *negative*, and is one *greater* than the number of ciphers before the first significant figure of the fraction.

Thus, the characteristic of the logarithm of $\frac{1}{10}$ or .1 is -1; that of $\frac{1}{100}$, or .01, is -2; that of $\frac{1}{1000}$, or .001, is -3, etc. (Art. 256.)

The logarithm of .2 is -1 with a decimal added to it; that of .05 is -2 with a decimal added to it, etc.

NOTE.—It should be observed that the *characteristic only* is *negative*, while the mantissa, or *decimal* part, is always *positive*. To indicate this, the sign $-$ is placed *over* the characteristic, instead of *before* it.

Thus, the logarithm of .2 is $\bar{1}.30103$,

" " " .05 is $\bar{2}.69897$, etc.

445. The ***Decimal Part*** of the logarithm of any number is the same as the logarithm of the number multiplied or divided by 10, 100, 1000, etc.

Thus, the logarithm of 1876 is 3.27325; of 18760 is 4.27325, etc.

TABLES OF LOGARITHMS.

446. A ***Table of Logarithms*** is one which contains the logarithms of all numbers between given limits.

447. The Table found on the following pages gives the mantissas of common logarithms to five decimal places for all numbers from 1 to 1000, inclusive.

The *characteristics* are omitted, and must be supplied by inspection. (Arts. 443, 444.)

NOTES.—1. The first decimal figure in column 0 is often the same for several successive numbers, but is printed only once, and is understood to belong to each of the blank places below it.

2. The character (•) shows that the figure belonging to the place it occupies has changed from 9 to 0, and through the rest of this line the first figure of the mantissa stands in the next line below.

---

444. What is the characteristic of the logarithm of a decimal? 445. What is the effect upon the decimal part of the log. of a number, if the number is multiplied or divided by 10, 100, 1000, etc. 446. What is a table of logarithms?

448. To Find the *Logarithm* of any Number from 1 to 10.

RULE.—*Look for the given number in the first line of the table; its logarithm will be found directly below it.*

1. Find the logarithm of 7. *Ans.* 0.84510.
2. Find the logarithm of 9. *Ans.* 0.95424.

449. To Find the *Logarithm* of any Number from 10 to 1000, inclusive.

RULE.—*Look in the column marked N for the first two figures of the given number, and for the third at the head of one of the other columns.*

Under this third fiqure, and opposite the first two, will be found the last decimal figures of the logarithm. The first one is found in the column marked 0.

To this decimal prefix the proper characteristic. (Art. 443.)

NOTE.—If the number contains 4 or more figures, multiply the tabular difference by the remaining figures, and rejecting from the right of the product as many figures as you multiply by, add the rest to the log. of the first 3 figures.

3. Find the logarithm of 108. *Ans.* 2.03342.
4. Find the logarithm of 176. *Ans.* 2.24551.
5. Find the logarithm of 1999. *Ans.* 3.30085.

450. To Find the Logarithm of a *Decimal Fraction.*

RULE.—*Take out the logarithm of a whole number consisting of the same figures, and prefix to it the proper negative characteristic.* (Art. 444.)

NOTE.—If the number consists of an *integer* and a *decimal*, find the logarithm in the same manner as if *all* the figures were integers, and prefix the characteristic which belongs to the *integral* part. (Art. 443.)

6. What is the log. of 0.95? *Ans.* $\bar{1}.97772$.
7. What is the log. of 0.0125? *Ans.* $\bar{2}.09691$.
8. What is the log. of 0.0075? *Ans.* $\bar{3}.87506$.
9. What is the log. of 16.45? *Ans.* 1.21616.
10. What is the log. of 185.3? *Ans.* 2.26787.

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448. How find the logarithm of a number from 1 to 10? 449. From 10 to 1000? 450. How find the log. of a decimal? *Note.* Of an integer and a decimal?

451. To Find the *Number* belonging to a given Logarithm.

RULE.—*Look for the decimal figures of the given logarithm in the table under the column marked* 0; *and if all of them are not found in that column, look in the other columns on the right till you find them exactly, or very nearly; directly opposite, in the column marked N, will be found the first two figures, and at the top, over the logarithm, the third figure of the given number.*

Make this number correspond to the characteristic of the given logarithm, by pointing off decimals, or by adding ciphers, if necessary, and it will be the number required.

NOTE.—If the characteristic of a logarithm is *negative*, the number belonging to it is a *fraction*, and as many ciphers must be prefixed to the number found in the table, as there are *units* in the characteristic *less* 1. (Art. 444.)

452. When the *Decimal Part* of the given Logarithm is not exactly, or very nearly, found in the Table.

RULE.—*From the given logarithm subtract the next less logarithm found in the tables; annex ciphers to the remainder, and divide it by the tabular difference (marked D) as far as necessary.*

To the number belonging to the less logarithm annex the quotient, and make the number thus produced correspond to the characteristic of the given logarithm, as above.

NOTE.—For every cipher annexed to the remainder, either a *significant* figure or a *cipher* must be put in the quotient.

11. What number belongs to 2.17231? *Ans.* 148.7.
12. What number belongs to 1.25261? *Ans.* 17.89.
13. What number belongs to 3.27715? *Ans.* 1893.
14. What number belongs to 2.30963? *Ans.* 204.
15. What number belongs to 4.29797? *Ans.* 19858.29.
16. What number belongs to $\bar{1}.14488$? *Ans.* 0.1396.
17. What number belongs to $\bar{2}.29136$? *Ans.* 0.01956.
18. What number belongs to $\bar{3}.30928$? *Ans.* 0.002038.

451. How find the number belonging to a logarithm?

453. Computations by logarithms are based upon the following principles:

1°. *The sum of the logarithms of two numbers is equal to the logarithm of their product.*

Let a and c denote any two numbers, m and n their logarithms, and b the base.

Then	$b^m = a$
And	$b^n = c$
Multiplying,	$b^{m+n} = ac.$

2°. *The logarithm of the dividend diminished by the logarithm of the divisor is equal to the logarithm of the quotient of the two numbers.*

Let a and c denote any two numbers, m and n their logarithms, and b the base.

Then	$b^m = a$
And	$b^n = c$
Dividing,	$b^{m-n} = a \div c.$

454. To *Multiply* by Logarithms.

1. Required the product of 35 by 23.

SOLUTION.—The log. of 35 =	1.54407	
" " " 23 =	1.36173	
Adding,	2.90580.	(Art. 453, Prin. 1.)

The number belonging is 805, *Ans.* Hence, the

RULE—*Add the logarithms of the factors; the sum will be the logarithm of the product.*

NOTES.—1. If the sum of the decimal parts exceeds 9, add the *tens* figure to the characteristic.

2. If either or all the characteristics are *negative*, they must be added according to Art. 65. But as the mantissa is always *positive*, that which is carried from the mantissa to the characteristic must be considered *positive*.

2. What is the product of 109.3 by 14.17?

3. What is the product of 1.465 by 1.347?

4. What is the product of .074 by 1500?

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453. Upon what two principles are computations by logarithms based? 454. How multiply by logarithms?

455. To *Divide* by Logarithms.

5. Required the quotient of 120 by 15.

SOLUTION.—The log. of 120 = 2.07918
" " " 15 = 1.17609
" " " quotient = 0.90309. *Ans.* 8. Hence, the

RULE.—*From the logarithm of the dividend subtract the logarithm of the divisor; the difference will be the logarithm of the quotient.* (Art. 453, Prin. 2.)

NOTES.—1. When either of the characteristics is *negative*, or when the lower one is greater than the one above it, change the sign of the subtrahend, and proceed as in addition.

2. When 1 is carried from the mantissa to the characteristic, it must be considered *positive*, and be added to the characteristic *before* the sign is changed.

6. What is the quotient of 12.48 by 0.16?
7. What is the quotient of .045 by 1.20?
8. What is the quotient of 1.381 by .096?

456. *Negative* quantities are divided in the same manner as *positive* quantities.

If the sign of the divisor is the same as that of the dividend, prefix the sign + to the quotient; but if different, prefix the sign —.

9. Divide —128 by —47.
10. Divide —186 by —0.064.
11. Divide —0.156 by —0.86.
12. Divide —0.194 by 0.042.

457. To *Involve* a Number by Logarithms.

Multiplication by logarithms is performed by *addition*. (Art. 453.) Therefore, if the logarithm of a quantity is added *to itself once*, the result will be the logarithm of the *second power* of that quantity; if added to itself *twice*, the result will be the *third* power of that quantity, and so on. Hence,

RULE.—*Multiply the logarithm of the number by the exponent of the required power.*

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455. How divide by them? 457. How involve a number by logarithms?

NOTES.—1. This rule depends upon the principle that logarithms are the *exponents* of *powers* and *roots*, and a power or root is involved by multiplying its *index* into the *index* of the power required.

2. In this rule, whatever is carried from the mantissa to the characteristic is *positive*, whether the index itself is *positive* or *negative*.

13. What is the cube of 1.246.

SOLUTION.—The log. of 1.246 is	0.09551	
Index of the required power is	3	
Log. of power is	0.28653.	*Ans.* 1.93435.

14. What is the fourth power of .135?

15. What is the tenth power of 1.42?

16. What is the twenty-fifth power of 1.234?

458. To *Extract the Root* of a Number by Logarithms.

A quantity is resolved into any number of *equal factors*, by dividing its index into as many *equal parts*. (Art. 281.) Hence, the

RULE.—*Divide the logarithm of the number by the index of the required root.*

NOTE.—This rule depends upon the principle that the *root* of a quantity is found by *dividing the exponent* by the number expressing the required root.

17. What is the square root of 1.69?

SOLUTION.—The log. of 1.69 is	0.22789	
The index is 2,	2) .22789	
Logarithm of root,	0.11394.	*Ans.* 1.3.

18. What is the cube root of 143.2?

19. What is the sixth root of 1.62?

20. What is the eighth root of 1549?

21. What is the tenth root of 1876?

459. If the characteristic of the logarithm is *negative*, and cannot be *divided* by the *index* of the required root without a *remainder*, make it *positive* by adding to the *characteristic* such a *negative* number as will make it *exactly divisible* by the *divisor*, and *prefix* an equal *positive* number to the decimal part of the logarithm.

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458. How extract the root?

22. It is required to find the cube root of .0164.

SOLUTION.—The log. of .0164 is $\bar{2}.21484$.

Preparing the log., $3\,)\,\bar{3}+1.21484$

$\bar{1}.40494$. *Ans.* 0.25406+.

23. What is the sixth root of .001624?

24. What is the seventh root of .01449?

25. What is the eighth root of .0001236?

460. To Calculate *Compound Interest* by Logarithms.

RULE.—*Find the amount of 1 dollar for 1 year; multiply its logarithm by the number of years, and to the product add the logarithm of the principal. The sum will be the logarithm of the amount for the given time.*

From the amount subtract the principal, and the remainder will be the interest.

NOTES.—1. If the interest becomes due *half yearly* or *quarterly*, find the amount of one dollar for the half year or quarter, and multiply the logarithm by the number of half years or quarters in the time.

2. This rule is based upon the principle that the several *amounts* in compound interest form a *geometrical series*, of which the principal is the first term, the amount of $1 for 1 year the ratio, and the number of years + 1 the number of terms

26. What is the amount of $1565 for 40 years, at 6 per cent compound interest?

SOLUTION.—The amt. of $1 for 1 year is $1.06; its log.,	0.02531
The number of years,	40
Product,	1.01240
The given principal, $1565; its log.,	3.19453
Ans. $16103.78.	4.20693

27. What is the amount of $1500, at 7 per cent compound interest, for 4 years? *Ans.* $1966.05.

28. What is the amount of $370, at 5 per cent compound interest for 33 years? *Ans.* $1851.27+.

460. How calculate compound interest by logarithms?

TABLE OF COMMON LOGARITHMS.

N.	0	1	2	3	4	5	6	7	8	9	D.
	.00000	.00000	.30103	.47712	.60206	.69897	.77815	.84510	.90309	.95424	
10	.00000	0432	0860	1284	1703	2119	2531	2938	3342	3743	416
11	4139	4532	4922	5308	5691	6070	6446	6819	7188	7555	379
12	7918	8279	8636	8991	9342	9691	◆037	◆380	◆721	1059	349
13	.11394	1727	2057	2385	2711	3033	3354	3672	3988	4302	322
14	4613	4922	5229	5534	5836	6137	6435	6732	7026	7319	301
15	7609	7898	8184	8469	8752	9033	9313	9590	9866	◆140	281
16	.20412	0683	0952	1219	1484	1748	2011	2272	2531	2789	264
17	3045	3300	3553	3805	4055	4304	4551	4797	5042	5285	249
18	5527	5768	6007	6245	6482	6717	6951	7184	7416	7646	235
19	7875	8103	8330	8556	8780	9003	9226	9447	9667	9885	223
20	.30103	0320	0535	0750	0963	1175	1387	1597	1806	2015	212
21	2222	2428	2634	2838	3041	3244	3445	3646	3846	4044	203
22	4242	4439	4635	4831	5025	5218	5411	5603	5794	5984	193
23	6173	6361	6549	6736	6922	7107	7291	7475	7658	7840	185
24	8021	8202	8382	8561	8739	8917	9094	9270	9445	9620	178
25	9794	9967	◆140	◆312	◆483	◆654	◆824	◆993	1162	1330	171
26	.41497	1664	1830	1996	2160	2325	2488	2651	2814	2975	165
27	3136	3297	3457	3616	3775	3933	4091	4248	4405	4560	158
28	4716	4871	5025	5179	5332	5485	5637	5788	5939	6090	153
29	6240	6389	6538	6687	6835	6982	7129	7276	7422	7567	147
30	.47712	7857	8001	8144	8287	8430	8572	8714	8855	8996	143
31	9136	9276	9416	9554	9693	9831	9969	◆106	◆243	◆379	138
32	.50515	0651	0786	0920	1055	1188	1322	1455	1587	1720	133
33	1851	1983	2114	2244	2375	2505	2634	2763	2892	3020	130
34	3148	3275	3403	3529	3656	3782	3908	4033	4158	4283	126
35	4407	4531	4654	4778	4900	5023	5145	5267	5388	5509	123
36	5630	5751	5871	5991	6110	6229	6348	6467	6585	6703	119
37	6820	6937	7054	7171	7287	7403	7519	7634	7749	7864	116
38	7978	8093	8206	8320	8433	8546	8659	8771	8883	8995	113
39	9107	9218	9329	9439	9550	9660	9770	9879	9988	◆097	110
40	.60206	0314	0423	0531	0638	0746	0853	0959	1066	1172	108
41	1278	1384	1490	1595	1700	1805	1909	2014	2118	2221	105
42	2325	2428	2531	2634	2737	2839	2941	3043	3144	3246	102
43	3347	3448	3548	3649	3749	3849	3949	4048	4147	4247	100
44	4345	4444	4542	4640	4738	4836	4934	5031	5128	5225	98
45	5321	5418	5514	5610	5706	5801	5897	5992	6087	6181	95
46	6276	6370	6464	6558	6652	6745	6839	6932	7025	7117	93
47	7210	7302	7394	7486	7578	7669	7761	7852	7943	8034	91
48	8124	8215	8305	8395	8485	8574	8664	8753	8842	8931	89
49	9020	9108	9197	9285	9373	9461	9548	9636	9723	9810	88
N.	0	1	2	3	4	5	6	7	8	9	D.

NOTE.—When the stock is *above* or *below* par, the *premium* or *discount* must be added to or subtracted from its par value to give the cost.

30. What per cent interest does a man receive on an investment of $5000 in the Bank of Commerce, its dividends being 10 per cent, and the shares 5 per cent above par?

SOLUTION.—The premium on the stock = $5000 × .05 = $250. Therefore, the cost = $5000 + $250 = $5250.

Again, the dividend on stock = $5000 × .10 = $500. Therefore, $500 ÷ $5250 = $9\frac{11}{21}$ per cent, *Ans.*

31. A invested $6000 in New York 6 per cent bonds, at 3 per cent premium. What per cent did he receive on his investment?

32. A man lays out $1000 in Alabama 10 per cents, at a discount of 20 per cent. What per cent did he receive on his investment?

33. What per cent will a man receive on 50 shares of Pennsylvania Railroad stock, the premium being 4 per ct., and the dividend 10 per cent?

34. Which are preferable, Massachusetts 6 per cent bonds at par, or Ohio 8 per cent bonds at 2 per cent premium?

496. To Find the *Amount* of a given Remittance which a Factor can Invest, and Reserve a Specified Per Cent for his Commission.

Let s denote the sum remitted, and r the per cent commission.

The sum remitted includes both the sum invested and the commission. Now $1 remitted is 100 per cent, or once itself; and adding the per cent to it, we have $1 + r$, the cost of $1 invested. Therefore, $s \div (1 + r)$ must be the amount invested.

Putting a for the amount invested, we have the

FORMULA. $$a = \frac{s}{1 + r}.$$

RULE.—*Divide the remittance by 1 plus the per cent commission; the quotient will be the amount to invest.*

35. A clergyman remitted to his agent \$2500 to purchase books. After deducting 4 per cent commission, how much does he lay out in books?

SOLUTION. $a = \frac{s}{1+r} = \frac{\$2500}{1.04} = \$2403.85$, *Ans.*

36. A gentleman remitted \$25000 to a broker, to be invested in stocks. After deducting $1\frac{1}{2}$ per cent, how much did he invest, and what was his commission?

SINKING FUNDS.

497. ***Sinking Funds*** are sums of money set apart or deposited annually, for the payment of public debts, and for other purposes.

CASE I

498. To Find the *Amount* of an Annual Deposit at Compound Interest, the Rate and Time being given.

Let s denote the annual deposit or sum set apart, r the rate per cent, n the number of years, and a the amount required.

Since the same sum is deposited at the end of each year, and put at compound interest, it follows that the deposit at the end of the

1st year $= s$
2d " $= s + s(1+r)$
3d " $= s + s(1+r) + s(1+r)^2$
nth " $= s + s(1+r) + s(1+r)^2 \ldots + s(1+r)^{n-1}$,

forming a geometrical series; the annual deposit being the first term, the amount of \$1 for 1 year the ratio, the number of years the number of terms, and the annual deposit multiplied by the amount of \$1 for 1 year, raised to that power whose index is 1 less than the number of years, the last term; and the amount is equal to the sum of the series. (Art. 402.) Hence, we have the

FORMULA. $$a = \frac{(1+r)^n - 1}{r} s.$$

RULE.—*Multiply the amount of \$1 annual deposit for the given time and rate by the given annual deposit; the product will be the amount required.*

37. A clerk annually deposited \$150 in a savings bank which pays 6 per cent compound interest. What amount will be due him in 5 years?

SOLUTION. $a = \frac{(1+r)^n - 1}{r} s = \frac{(1.06)^5 - 1}{.06}$ \$150 = \$845.75, *Ans.*

38. A man agrees to give \$300 annually to build a church. What will his subscription amount to in 4 years, at 7 per cent compound interest?

39. If a teacher lays up \$500 annually, and puts it at 5 per cent compound interest for 10 years, how much will he be worth?

CASE II.

499. To Find the *Annual Deposit* required to produce a given Amount at Compound Interest, the Rate and Time being given.

By the formula in the preceding article, we have

$$\frac{(1+r)^n - 1}{r} s = a.$$

Dividing by coefficient of s, we have the

$$\text{FORMULA.} \qquad s = a \div \frac{(1 + r)^n - 1}{r}.$$

RULE.—*Divide the amount to be raised by the amount of \$1 annual deposit for the given time and rate; the quotient will be the annual deposit required.*

NOTE.—To cancel the debt at maturity, the sum set apart as a sinking fund is supposed to be put at compound interest for the given time and rate.

40. A father promises to gives his daughter \$5000 as a wedding present. Suppose the event to occur in 5 years, what sum must he annually deposit in a Trust Company, at 5 per cent compound interest, to meet his engagement?

SOLUTION. $s = a \div \frac{(1 + r)^n - 1}{r} = \$5000 \div \frac{(1.05)^5 - 1}{.05} = \frac{\$250}{.276} =$ \$905.80, *Ans.*

41. A man having lost his patrimony of $20000, wishes to know how much must be annually deposited at 10 per cent, to recover it in 5 years?

42. A county borrows $30000, at 6 per cent compound interest, to build a court-house; what sum must be set apart annually as a sinking fund to cancel the debt in 10 years?

ANNUITIES.

500. ***Annuities*** are sums of money payable annually, or at regular intervals of time. They are computed according to the principles of compound interest.

CASE I.

501. To Find the *Amount* of an Unpaid Annuity at Compound Interest, the Time and Rate per cent being given.

Let a be the annuity, $1 + r$ the amount of $1 for 1 year, and n the number of years.

The amount due at the end of the

1st year $= a$,

2d " $= a + a(1+r)$

3d " $= a + a(1+r) + a(1+r)^2$,

4th " $= a + a(1+r) + a(1+r)^2 + a(1+r)^3$,

nth year $= a + a(1+r) + a(1+r)^2 + a(1+r)^3 \ldots a(1+r)^{n-1}$.

forming a geometrical progression, the annuity being the first term, the amount of $1 for 1 year the ratio, the number of years the number of terms, and the annuity multiplied by the amount of $1 for 1 year raised to that power whose index is 1 less than the number of years, the last term. Therefore, the amount is equal to the sum of the series.

Putting S for the amount (Art. 498), we have the

FORMULA. $$S = \frac{(1+r)^n - 1}{r}a.$$

RULE.—*Multiply the amount of $1 annuity, for the given time and rate, by the given annuity.*

NOTE.—Logarithms may be used to advantage in some of the following examples.

43. What is due on an annuity of \$650, unpaid for 4 years, at 7 per cent compound interest?

SOLUTION. $S = \frac{(1+r)^n - 1}{r} a = \frac{(1.07)^4 - 1}{.07}$ \$650 = \$2886, *Ans.*

44. An annual pension of \$880 was unpaid for 6 years; what did it amount to at 6 per cent compound interest?

45. An annual tax of \$340 was unpaid for 7 years; what was due on it at 5 per cent compound interest?

CASE II.

502. To Find the *Present Worth* of an Annuity at Compound Interest, the Time of Continuance and the Rate being given.

Let P denote the present worth; then the amount of P in n years will be equal to the amount of the annuity for the same time. Therefore,

$$P(1+r)^n = \frac{(1+r)^n - 1}{r} a.$$

Dividing each member by $(1+r)^n$ (Art. 279), we have the

FORMULA. $$P = \frac{1 - (1+r)^{-n}}{r} a.$$

NOTE.—In applying the formula, the *negative* exponent may be made positive by *transferring* the quantity which it affects from the *numerator* to the denominator (Art. 279).

Thus, $$P = \frac{1-(1+r)^{-n}}{r} a = \frac{1 - \frac{1}{(1+r)^n}}{r} a.$$

RULE.—*Multiply the present worth of an annuity of \$1 for the given time by the given annuity.*

46. What is the present worth of an annuity of \$375 for 6 years, at 7 per cent compound interest?

SOLUTION. $P = \frac{1-(1+r)^{-n}}{r} a = \frac{1-(1.07)^{-6}}{.07}$ \$375 $= \frac{1 - .66}{.07}$ \$375 = \$1785.71, *Ans.*

47. What is the present worth of an annual pension of \$525 for 5 years, at 4 per cent compound interest?

CASE III.

503. To Find the *Present Worth* of a Perpetual Annuity, the Rate being given.

Let n denote infinity, then reducing the formula in Art. 502, we have this

FORMULA. $P = \frac{a}{r}$. (Art. 435.)

RULE.—*Divide the annuity by the interest of $1 for 1 year, at the given rate.*

48. What is the present worth of a perpetual scholarship that pays $150 annually, at 7 per cent compound interest?

SOLUTION. $P = \frac{a}{r} = \frac{\$150}{.07} = \$2142.86$, *Ans.*

49. What is the present worth of a perpetual ground rent of $850 a year, at 6 per cent?

CASE IV.

504. To Find the *Present Worth* of an Annuity, commencing in a given Number of Years, the Rate and Time of Continuance being given.

Let n be the number of years before it will commence, and N the number of years it is to continue. Then,

$$P = a\frac{1-(1+r)^{-(n+N)}}{r} - a\frac{1-(1+r)^{-n}}{r}.$$

Performing the subtraction indicated, we have the

FORMULA. $P = \frac{a}{r}[(1+r)^{-n} - (1+r)^{-n-N}]$.

RULE.—*Find the present worth of the given annuity to the time it terminates; from this subtract its present worth to the time it commences.*

50. What is the present worth of an annuity of $600, to commence in 4 years and to continue 12 years, at 7 per cent interest?

SOLUTION. $P = \frac{a}{r}[(1+r)^{-n} - (1+r)^{-n-N}]$,

$$P = \frac{\$600}{.07}[(1.07)^{-4} - (1.07)^{-4-12}],$$

$$P = \frac{\$600 \times .4242}{.07} = \$3636, \textit{ Ans.}$$

51. A father left an annual rent of \$2500 to his son for 6 years, and the reversion of it to his daughter for 12 years. What is the present worth of her legacy at 6 per cent interest?

CASE V.

505. To Find the *Annuity*, the Present Worth, the Time, and Rate being given.

By the formula in Article 502,

$$P = \frac{1 - (1 + r)^{-n}}{r} a.$$

Dividing by the coefficient of a, we have the

FORMULA. $$a = \frac{Pr}{1 - (1 + r)^{-n}}.$$

RULE.—*Divide the present worth by the present worth of an annuity of \$1 for the given time and rate.*

52. The present worth of a pension, to continue 20 years at 6 per cent interest, is \$668. Required the pension.

SOLUTION. $$a = \frac{Pr}{1-(1+r)^{-n}} = \frac{\$668 \times .06}{1-(1.06)^{-20}} = \frac{\$668 \times .06}{.6882} = \$58.23, \textit{Ans.}$$

53. The present worth of an annuity, to continue 30 years at 5 per cent interest, is \$3840. What is the annuity?

NOTE.—The process of constructing formulas or rules, it will be seen, is based upon the principles of generalization combined with those of algebraic notation. The student will find it a profitable exercise to form others applicable to different classes of problems.

CHAPTER XXI.

DISCUSSION OF PROBLEMS.

506. The ***Discussion*** of a Problem consists in assigning all the different values possible to the arbitrary quantities which it contains, and interpreting the results.

507. An ***Arbitrary Quantity*** is one to which any value may be assigned at pleasure.

PROBLEM.—If b is subtracted from a, by what number must the remainder be multiplied that the product may be equal to c?

Let $\quad x =$ the number.

Then $\quad (a-b)x = c.$

Therefore, $\quad x = \dfrac{c}{a-b}.$

508. The result thus obtained may have five different forms, depending on the relative values of a, b, and c. To represent these forms, let m denote the multiplier.

I. Suppose a is *greater* than b. In this case $a-b$ is *positive*, and c being *positive*, the quotient is *positive*. (Art. 112.) Consequently, the required multiplier must be *positive*, and the value of x will be of the form of $+m$.

II. Suppose a is *less* than b. In this case $a-b$ is *negative*, and c divided by $a-b$ is *negative*. (Art. 112.) Hence, the required multiplier must be *negative*, and the value of x is of the form of $-m$.

III. Suppose a is *equal* to b. In this case $a-b=0$. Therefore, the value of x is of the form of $\frac{m}{0}$, or $x = \frac{c}{0} = \infty$. (Art. 434.)

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506. In what does the discussion of problems consist? 507. What is an arbitrary quantity?

IV. Suppose c is 0, and a is either *greater* or *less* than b. In this case the value of x has the form $\frac{0}{m}$, or $x = 0$.

V. Suppose c equals 0, and a equals b. In this case the value of x has the form $\frac{0}{0}$.

NOTE.—The student can easily test these principles by substituting numbers for a, b, and c.

509. The Discussion of Problems may be further illustrated by the solution of the celebrated

PROBLEM OF THE COURIERS.*

Two couriers A and B, were traveling along the same road in the same direction, from C toward Q; A going at the rate of m miles an hour, and B n miles an hour. At 12 o'clock A was at a certain point P; and B d miles in advance of A, in the direction of Q. When and where were they together?

This problem is general: we do not know from the statement whether the couriers were together before or after 12 o'clock, nor whether the place of meeting was on the right or the left of P.

Suppose the required time to be *after* 12 o'clock. Then the time after 12 is *positive*, and the time before 12 is *negative;* also, the distance reckoned from P toward Q is *positive*, and from P toward C is *negative*.

Let t = time of meeting in hours after 12 o'clock; then mt = distance from P to the point of meeting.

Since A traveled at the rate of m miles an hour, and B n miles an hour, we have

$$mt = \text{the distance A traveled.}$$

And $nt =$ " " B "

Again, since A and B were d miles apart at 12 o'clock,

$$mt - nt = d.$$

Factoring and dividing we have the

* Originally proposed by Clairaut, an eminent French mathematician, born in 1713.

FORMULA. $t = \frac{d}{m - n}$.

The problem may now be discussed in relation to the time t, and the distance mt, the two unknown elements.

I. Suppose $m > n$.

Upon this supposition the values of t and mt will both be positive; because their denominator $m - n$ is positive. Now since t is *positive*, it is evident the two couriers came together *after* 12 o'clock; and as mt is *positive*, the point of meeting was somewhere on the right of P.

These conclusions agree with each other, and correspond to the conditions of the problem. For, the supposition that $m > n$ implies that A was traveling faster than B. A would therefore gain upon B, and overtake him some time after 12 o'clock, and at a point in the direction of Q.

Let $d = 24$ miles, $m = 8$ miles, and $n = 6$ miles.

By the formula, $t = \frac{d}{m - n} = \frac{24}{8 - 6} = 12$ hours.

$mt = 8 \times 12 = 96$ miles A traveled.

$nt = 6 \times 12 = 72$ " B "

Now, $96 - 72 = 24$ m. their distance apart at noon, as given above.

These values show that the couriers were together in 12 hours past noon, or at midnight, and at a point Q, 96 miles from P and 72 miles from d.

II. Suppose $m < n$.

Then in the formula, the denominator $m - n$ is negative, therefore both t and mt are negative.

Hence, both t and mt must be taken in a sense contrary to that which they had in supposition (I), where they were positive; that is, the time the couriers were together was *before* 12 o'clock, and the place of meeting on the *left* of P.

This interpretation is also in accordance with the conditions of the problem under the present supposition. For, if $m < n$, then B was traveling faster than A; and as B was in advance of A at 12 o'clock, he must have passed A before that time, somewhere on the left of P, in the direction of C.

Let $d = 24$ miles, $m = 5$ miles, and $n = 8$ miles.

By the formula, $t = \frac{d}{m - n} = \frac{24}{5 - 8} = -8$ hours.

And $mt = 5 \times -8 = -40$ miles A traveled.

" $nt = 8 \times -8 = -64$ " B "

These values show that the couriers were together 8 hours before noon, or at 4 o'clock A. M., and at a point C, 40 miles from P and 64 miles from d.

III. Suppose $m = n$.

Upon this supposition we have $m - n = 0$, and

$$t = \frac{d}{0} = \infty, \quad \text{also} \quad mt = \frac{md}{0} = \infty.$$

According to these results, t the time to elapse before the couriers are together, is infinity (Art. 434): consequently they can never be together. In like manner mt, the distance from P of the supposed point of meeting, is infinity; hence. there can be no such point.

This interpretation agrees with the supposed conditions of the problem. For, at 12 o'clock the two couriers were d miles apart, and if $m = n$ they were traveling at equal rates, and therefore could never meet.

IV. Suppose $d = 0$, and m either greater or less than n.

We then have $t = \dfrac{0}{m - n} = 0$, and $mt = 0$.

That is, both the time and distance are *nothing*. These results show that the couriers were together at 12 o'clock at the point P, and at no other time or place.

This interpretation is confirmed by the conditions of the problem. For, if $d = 0$, then at 12 o'clock B must have been with A at the point P. And if m is greater than n, or m is less than n, the couriers were traveling at different rates, and must either approach or recede from each other at all times, except at the moment of passing; therefore they can be together only at a single point.

V. Suppose $d = 0$, and $m = n$.

Then we have $\quad t = \dfrac{0}{0}, \quad$ and $\quad mt = \dfrac{0}{0}$.

These results must be interpreted to mean that the time and the distance may be anything whatever, and that the couriers must be together at all times, and at any distance from P.

This conclusion also corresponds to the conditions of the problem. For, if $d = 0$, the couriers were together at 12 o'clock, and if $m = n$, they were traveling at equal rates, and therefore would never part.

IMAGINARY QUANTITIES.

510. An ***Imaginary Quantity*** is an indicated *even* root of a *negative* quantity; as, $\sqrt{-1}$, $\sqrt{-a}$, $\sqrt[4]{-7}$.

NOTES.—1. Imaginary quantities are a species of *radicals*, and are called *imaginary*, because they denote operations which it is impossible to perform. (Art. 294.)

2. Though the operations indicated are in themselves *impossible*, these imaginary expressions are often useful in mathematical analyses, and when subjected to certain modifications, lead to important results.

511. Imaginary quantities are *added* and *subtracted* like other radicals. (Arts. 310, 311.)

But to *multiply* and *divide* them, some modifications in the rules of radicals are required. (Arts. 312, 313.)

512. To Prepare an *Imaginary Quantity* for Multiplication and Division.

RULE.—*Resolve the given quantity into two factors, one of which is a real quantity, and the other the imaginary expression* $\sqrt{-1}$.

NOTES.—1. This modification is based upon the principle that any *negative* quantity may be regarded as the product of two quantities, one of which is -1. Thus, $-a = a \times -1$; $-b^2 = b^2 \times -1$.

2. The real factor is often called the coefficient of the imaginary expression, $\sqrt{-1}$.

1. Multiply $\sqrt{-a}$ by $\sqrt{-b}$.

SOLUTION. $\sqrt{-a} = \sqrt{a} \times \sqrt{-1}$, and $\sqrt{-b} = \sqrt{b} \times \sqrt{-1}$. Now $\sqrt{a} \times \sqrt{-1} \times \sqrt{b} \times \sqrt{-1} = \sqrt{ab} \times -1 = -\sqrt{ab}$, *Ans.*

2. Multiply $+\sqrt{-x}$ by $-\sqrt{-y}$.

3. Multiply $\sqrt{-9}$ by $\sqrt{-4}$.

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510. What are imaginary quantities? 511. How added and subtracted? 512. How prepare them for multiplication and division?

513. It will be seen from the preceding examples:

First. That the product of two imaginary quantities is a real quantity.

Second. That the sign before the product is the *opposite* of that required by the common rule for signs. (Art. 92.)

For, while the sign to be prefixed to an even root is ambiguous, this ambiguity is removed when we know whether the quantity whose root is to be taken has been produced from positive or negative quantities. (Art. 326.)

4. Multiply $\sqrt{-2}$ by $\sqrt{18}$,
5. Multiply $\sqrt{-x}$ by $\sqrt{y}$.

NOTE.—1. From these examples it will be seen that the product of a *real* quantity and an *imaginary* expression, is itself imaginary.

6. Divide $\sqrt{-x}$ by $\sqrt{-y}$.

SOLUTION.—$\dfrac{\sqrt{-x}}{\sqrt{-y}} = \dfrac{\sqrt{x} \times \sqrt{-1}}{\sqrt{y} \times \sqrt{-1}} = \sqrt{\dfrac{x}{y}}$, *Ans.*

7. Divide $\sqrt{-x}$ by $\sqrt{-x}$,

NOTE.—2. Hence, the *quotient* of one imaginary quantity divided by another, is a *real* quantity; and the sign before the radical is the same as that prescribed by the rule. (Art. 92.)

8. Divide $\sqrt{-x}$ by $\sqrt{y}$,
9. Divide $\sqrt{x}$ by $\sqrt{-y}$,

NOTE.—3. Hence, the *quotient* of an imaginary quantity divided by a real one, is itself imaginary, and *vice versa.*

10. Divide $10\sqrt{-14}$ by $2\sqrt{-7}$,
11. Divide $c\sqrt{-1}$ by $d\sqrt{-1}$,

514. The development of the different powers of $\sqrt{-1}$.

12. $(\sqrt{-1})^2 = -1.$
13. $(\sqrt{-1})^3 = -\sqrt{-1}.$
14. $(\sqrt{-1})^4 = +1.$
15. $(\sqrt{-1})^5 = +\sqrt{-1}.$
16. $(\sqrt{-1})^6 = -1.$
17. $(\sqrt{-1})^7 = -\sqrt{-1}.$

Hence, *the even powers are alternately* -1 and $+1$, and the odd powers $-\sqrt{-1}$ and $+\sqrt{-1}$.

INDETERMINATE PROBLEMS.

515. An ***Indeterminate Problem*** is one which does not admit of a definite answer. (Art. 220.)

NOTE.—Among the more common indeterminate problems, are

1st. Those whose conditions are satisfied by different values of the same unknown quantity. (Art 220.)

2d. Those which produce identical equations. (Art. 200.)

3d. Those which have a less number of independent simultaneous equations than there are unknown quantities to be determined.

4th. Those whose conditions are inconsistent with each other.

1. Given the equation $x + y = 9$, to find the value of x.

SOLUTION.—Transposing, $x = 9 - y$, *Ans.* This result can be verified by assigning any values to x or y.

2. What number is that, $\frac{3}{4}$ of which minus 1 half of itself is equal to its 12th part plus its sixth part?

Let $x =$ the number,

Then $\frac{3x}{4} - \frac{x}{2} = \frac{x}{12} + \frac{x}{6}$

Clearing of fractions, etc., $9x = 9x$

Transposing and factoring, $(9-9)x = 0$

$\therefore \quad x = \frac{0}{0}.$

IMPOSSIBLE PROBLEMS.

516. An ***Impossible Problem*** is one, the conditions of which are *contradictory* or *impossible.*

1. Given $x + y = 10$, $x - y = 2$, and $xy = 38$.

OPERATION.

$$x + y = 10 \quad (1)$$
$$x - y = 2 \quad (2)$$
$$2x = 12$$
$$\therefore \quad x = 6$$
$$y = 4.$$

SOLUTION.—By combining equations (1) and (2), we find $x = 6$ and $y = 4$. Again, $x \times y = 6 \times 4 = 24$. But the third condition requires the product of x and y to be 38, which is impossible.

515. What is an indeterminate problem? 516. What is an impossible problem?

2. What number is that whose 5th part exceeds its 4th part by 15?

3. Divide 8 into two such parts that their product shall be 18.

NEGATIVE SOLUTIONS.

517. A ***Negative Solution*** is one whose result is a *minus* quantity.

518. An *odd root* of a quantity has the same sign as the quantity. An *even* root of a *positive* quantity is either *positive* or *negative,* both being numerically the same. (Art. 293.)

But the results of problems in Simple Equations, it is understood, are *positive;* when otherwise it is presumed there is an *error* in the data, which being corrected, the result will be positive.

1. A school-room is 30 feet long and 20 feet wide. How many feet must be added to its width that the room may contain 510 square feet?

SOLUTION.—Let $x =$ the number of feet,

Then $(20 + x)30 =$ area.

By conditions, $600 + 30x = 510$

Transposing, $30x = -90$

$\therefore \quad x = -3$ ft., *Ans.*

NOTES.—1. It will be observed that this is a problem in Simple Equations. The steps in the solution are legitimate and the result satisfies the conditions of the problem algebraically, but not arithmetically. Hence, the *negative result* indicates some *mistake* or *inconsistency* in the conditions of the problem.

If we subtract 3 ft. from its width, the result will be a *positive* quantity.

2. Were it asked how much must be added to the width that the room may contain 690 square feet, the result would be $+3$ feet.

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517. What is a negative solution?

3. In such cases, by changing some of the data, a similar problem may be easily found whose conditions are consistent with a possible result.

2. What number must be subtracted from 5 that the remainder may be 8?

SOLUTION.—Let $x =$ the number.
Then $5 - x = 8$
Transposing, $x = -3$, *Ans.*

3. A man at the time of his marriage was 36 years old and his wife 20 years. How many years before he was twice as old as his wife? *Ans.* -4 years.

HORNER'S METHOD OF APPROXIMATION.*

519. This method consists in transforming the given equation into another whose root shall be less than that of the given equation by the first figure of the root, and repeating the operation till the desired approximation is found.

The process may be illustrated in the following manner:

Let it be required to find the approximate value of x in the general equation,

$$Ax^3 + Bx^2 + Cx = D. \qquad (1)$$

Having found the first figure of the root by trial, let it be denoted by a, the second figure by b, the third by c, and so on.

Substituting a for x in equation (1), we have,

$$Aa^3 + Ba^2 + Ca = D, \text{ nearly.}$$

Factoring and dividing,

$$a = \frac{D}{C + Ba + Aa^2} \qquad (2)$$

* So called from the name of its author, an English mathematician, who communicated it to the Royal Society in 1819.

By putting y for the sum of all the figures of the root except the first, we have $x = a + y$, and substituting this value for x in equation (1), we have,

$$A(a+y)^3 + B(a+y)^2 + C(a+y) = D;$$

or $$A(a^3+3a^2y+3ay^2+y^3) + B(a^2+2ay+y^2) + C(a+y) = D.$$

or $$Aa^3+3Aa^2y+3Aay^2+Ay^3+Ba^2+2Bay+By^2+Ca+Cy = D.$$

Factoring and arranging the terms according to the powers of y, we obtain

$$Ay^3+(B+3Aa)y^2+(C+2Ba+3Aa^2)y = D-(Ca+Ba^2+Aa^3). \qquad (3)$$

To simplify this equation, let us denote the coefficient of y^2 by B', that of y by C', and the second member by D'; then,

$$Ay^3 + B'y^2 + C'y = D'. \qquad (4)$$

It will be seen that equation (4) has the same form as (1). It is the first transformed equation, and its root is less by a than the root of equation (1).

By repeating the operation, a second transformed equation may be obtained. Denoting the second figure in the root by b, and reducing as before, we find,

$$b = \frac{D'}{C' + B'b + Ab^2}. \qquad (5)$$

Putting z for the sum of all the remaining figures in the root, we have $y = b+z$; and substituting this value in equation (4), we obtain a new equation of the same general form, which may be written,

$$Az^3 + B''z^2 + C''z = D''. \qquad (6)$$

This process should be continued till the desired accuracy is attained. The first figure of the root is found by trial, the second figure from equation (5), and the remaining figures can be found from similar equations.

But it may be observed that the second member of equation (5) involves the quantity b, whose value is sought. That is, the value of b is given in terms of b, and that of c would be given in terms of c, and so on. For this reason, equations such as (5) might appear at first sight to be of little use in practice. This, however, is not the case; for after the root has been found to several decimal places, the value of the second and third terms, as $B'b+Ab^2$ and $B''c+Ac^2$ in the denominators, will be very small compared with C' and C'', conse-

quently as b is very nearly equal to D' divided by C', they may be neglected. Therefore the successive figures in the root may be approximately found by dividing D' by C', D'' by C'', and so on, regarding C', C'', etc., as approximate divisors.

In transforming equation (1) into (4), the second member D' and the coefficients C' and B' of the transformed equation may be thus obtained.

Multiplying the first coefficient A by a, the first figure of the root, and adding the product to B, the second coefficient, we have,

$$B + Aa \tag{7}$$

Again, multiplying this expression by a, and adding the product to C, the third coefficient, we have,

$$C + Ba + Aa^2. \tag{8}$$

Finally, multiplying these terms by a, and subtracting the product from D, we have

$$D - (Ca + Ba^2 + Aa^3) = D',$$

which is the same as the expression for D' in equation (4).

Now to obtain C', we return to the first coefficient, multiply it by a, add the product to expression (7), and thus have the sum

$$B + 2Aa, \tag{9}$$

which we multiply by a, and adding the product to expression (8) obtain,

$$C + 2Ba + 3Aa^2 = C',$$

which is the desired coefficient of y in equation (4).

Finally, to obtain B', we multiply the first coefficient by a, and add the product to expression (9), and thus obtain,

$$B + 3Aa = B'.$$

In this way the coefficients of the first transformed equation are discovered; and by a similar process the coefficients of the second, third, and of all subsequent transformed equations may be found.

520. This method of approximation is applicable to equations of every degree. For the solution of cubic equations, it may be summed up in the following

RULE.—I. *Detach the coefficients of the given equation, and denote them by A, B, C, and the second member by D. Find the first figure of the root by trial, and represent it by a,*

Multiply A by a, and add the product to B. Multiply the sum by a and add the product to C. Multiply this sum by a and subtract the product from D. The remainder is the first dividend, or D'.

II. *Multiply A by a and add the product to the last sum under B. Multiply this sum by a and add the product to the last sum under C. The result thus obtained is the first divisor, or C'.*

III. *Multiply A by a and add the product to the last sum under B. The result is the second coefficient, or B'.*

IV. *Divide the first dividend by the first divisor. The quotient is the second figure of the root, or b.*

V. *Proceed in like manner to find the subsequent figures of the root.*

Note.—1. In finding the second figure of the root, some allowance should be made for the terms in the divisor which are disregarded; otherwise the quotient will furnish a result too large to be subtracted from D'.

EXAMPLES.

1. Given $x^3 + 2x^2 + 3x = 24$, to find x.

SOLUTION.

A	B	C	D	$a\ b\ c$
1	+2	+3	= 24	$x = (2.08$, *Ans.*
	2	8	22	
	4	11	$2 = D'$	
	2	12	1.891712	
	6	$23 = C'$	$.108288 = D''$	
	2	.6464		
	$8 = B'$	23.6464		
	.08	.6528		
	8.08	$24.2992 = C''$		
	.08			
	8.16			
	.08			
	$9.24 = B''$			

NOTE.—2. In the following example, the last figures of the root are found by the contracted method of division of decimals, an expedient which may always be used to advantage after a few places of decimals have been obtained. (See Higher Arithmetic.)

2. Given $x^3 + 12x^2 - 18x = 216$, to find x.

SOLUTION.

A	B	C	D	abc
1	+12	−18	= 216	(4.24264+.
	4	+64	184	
	16	+46	32 = D'	
	4	80	26.168	
	20	126 = C'	5.832 = D''	
	4	4.8 4	5.468224	
	24 = B'	130.8 4	.363776 = D'''	
	24.2	4.8 8	275385	
	24.4	135.7 2 = C''	88391	
	24.6 = B''	.9 8 5 6	82615	
	24.64	136.7 0 5 6	5776	
	24.68	.9 8 7 2	5508	
		137.6|9|2|8 = C'''		

$x = 4.24264+$, *Ans.*

3. Given $x^3 + 3x^2 + 5x = 178$, to find x.

$x = 4.5388$, *Ans.*

4. Given $5x^3 + 9x^2 - 7x = 2200$, to find x.

$x = 7.1073536$, *Ans.*

5. Given $x^3 + x^2 + x = 100$, to find x.

$x = 4.264429+$, *Ans.*

TEST EXAMPLES FOR REVIEW.

1. Required the value of

$$6a + 4a \times 5 + 8a \div 2 - 3a + 12a \times 4.$$

2. Required the value of

$$(8x + 3x)5 + 4x + 7 - (5x + 9x) \div 7.$$

3. Required the value of

$$5ax - ab + 4cd - (2ax - 4ab + 2cd).$$

4. Required the value of

$$4bc + [3cd - (2xy - mn)5 + 3bc].$$

5. Show that subtracting a negative quantity is equivalent to adding a positive one.

6. Explain by an example why a positive quantity multiplied by a negative one produces a negative quantity?

7. Explain why a minus quantity multiplied by a minus quantity produces a positive quantity.

8. Given $\frac{2x}{3} - (x + 8) = \frac{48}{9} + \frac{12}{7} - 17\frac{5}{7}$, to find x.

9. Given $\frac{4x^2}{5} \div \frac{x}{5} + 2x = \frac{36}{3} \times \frac{31}{2}$, to find x.

10. Resolve $3b^2c - 6b^2c^2 - c^2d$ into two factors.

11. Resolve $3x^2y - 9x^2z - 18x^2yz$ into two factors.

12. Resolve $a^{2n} - b^{2n}$ into two factors.

13. Resolve $8a - 4$ into prime factors.

14. Resolve $a^4 - 1$ into prime factors.

15. Divide 31 into two such parts that 5 times one of them shall exceed 9 times the other by 1.

16. Make an algebraic formula by which any two numbers may be found, their sum and difference being given.

17. Two sportsmen at Creedmoor shoot alternately at a target; A hits the bull's-eye 2 out of 3 shots, and B 3 out of 4 shots; both together hit it 34 times. How many shots did each fire?

18. Find two quantities the product of which is a and the quotient b.

19. Reduce $\frac{a^2 - 1}{ab - b}$ to its lowest terms.

20. Reduce $\frac{a + b}{a^2 - b^2}$ to its lowest terms.

21. Resolve $9x^2y^2 + 12xyz + 4z^2$ into two factors.

22. Resolve $9b^2 - 6bc + c^2$ into two factors.

23. Make a formula by which the width of a rectangular surface may be found, the area and length being given?

24. A square tract of land contains $\frac{1}{4}$ as many acres as there are rods in the fence inclosing it. What is the length of the fence?

25. A student walked to the top of Mt. Washington at the rate of $1\frac{1}{2}$ miles an hour, and returned the same day at the rate of $4\frac{1}{2}$ miles an hour; the time occupied in traveling being 13 hours. How far did he walk?

26. Given $b - \frac{1 + x}{1 - x} = 0$, to find x.

27. Prove that the product of the sum and difference of two quantities, is equal to the difference of their squares.

28. Prove that the product of the sum of two quantities into a third quantity, is equal to the sum of their products.

29. Reduce $\frac{(x^2 - y^2)(x + y)}{(x^2 + 2xy + y^2)(x - y)}$ to its lowest terms.

30. Reduce $\frac{a^4 - b^4}{(a^2 - 2ab + b^2)(a^2 + b^2)}$ to its lowest terms.

31. Reduce $\frac{1 + a^2}{1 - a^4} - \frac{1 - a^2}{1 + a^2}$ to a single fraction having the least common denominator.

32. Find a number to which if its fourth and fifth part be added, the sum will exceed its sixth part by 154.

33. Two persons had equal sums of money; the first spent \$30, the second \$40: the former then had twice as much as the latter. What sum did each have at first?

34. A French privateer discovers a ship 24 kilometers distant, sailing at the rate of 8 kilometers an hour, and pursues her at the rate of 12 kilometers an hour. How long will the chase last?

35. Given $\frac{8x+3y}{7}=7$ and $\frac{7y-3x}{2}-y=0$, to find x and y.

36. Given $x=\frac{y-2}{7}+5$ and $4y-\frac{x+10}{3}=3$, to find x and y.

37. Make a rule to find when any two bodies moving toward each other will meet, the distance between them and the rate each moves being given?

38. A steamer whose speed in still water is 12 miles an hour, descended a river whose velocity is 4 miles an hour, and was gone 8 hours. How far did she go in the trip?

39. Find a fraction from which if 6 be subtracted from both its terms it becomes $\frac{1}{3}$, and if 6 be added to both, it becomes $\frac{3}{5}$.

40. Required two numbers whose sum is to the less as 8 is to 3, and the difference of whose squares is 49.

41. Given $10x+6y=76$, $4y-2z=8$, and $6x+8z=88$, to find x, y, and z.

42. Given $2x+3y+z=24$, $3x+y+2z=26$, and $x+2y+3z=34$, to find x, y, and z.

43. Three persons, A, B, and C, counting their money, found they had \$180. B said if his money were taken from the sum of the other two, the remainder would be \$60; C said if his were taken from the sum of the other two, the remainder would be $\frac{1}{2}$ of his money. How much money had each?

44. The fore-wheel of a steam-engine makes 40 revolutions more than the hind-wheel in going 240 meters, and the circumference of the latter is 3 meters greater than that of the former. What is the circumference of each?

45. A man has two cubical piles of wood; the side of one

is two feet longer than the side of the other, and the difference of their contents is 488 cubic feet. Required the side of each.

46. Required a formula by which the height of a rectangular solid may be found, the contents and base being given.

47. Divide 126 into two such parts that one shall be a multiple of 7, the other a multiple of 11.

48. A tailor paid $120 for French cloths; if he had bought 8 meters less for the same money, each meter would have cost 50 cents more. How many meters did he buy?

49. A shopkeeper paid $175 for 89 meters of silk. At what must he sell it a meter to make 25 per cent?

50. Make a formula to find the commercial discount, the marked price and the rate of discount being given.

51. A man pays $100 more for his carriage than for his horse, and the price of the former is to that of the latter as the price of the latter is to 50. What is the price of each?

52. Make a formula to find at what time the hour and minute hands of a watch are together between any two consecutive hours?

53. A father bequeathed 165 hectares of land to his two sons, so that the elder had 35 hectares more than the younger. How many hectares did each receive?

54. What number is that, the triple of which exceeds 40 by as much as its half is less than 51?

55. A butcher buys 6 sheep and 7 lambs for $71; and, at the same price, 4 sheep and 8 lambs for $64. What was the price of each?

56. At a certain election, 1425 persons voted, and the successful candidate had a majority of 271 votes. How many voted for each?

57. A's age is double B's, and B's is three times C's; the sum of all their ages is 150. What is the age of each?

58. Reduce the $\sqrt{243}$ to its simplest form.

59. Reduce $\sqrt{y^2 + ay^2}$ to its simplest form.

60. Reduce $x^{\frac{2}{3}}$ and $y^{\frac{1}{2}}$ to the common index $\frac{1}{6}$.

61. Reduce $3(a - b)$ to the form of the cube root.

62. A farmer sold 13 bushels of corn at a certain price; and afterward 17 bushels at the same rate, when he received \$3.60 more than at the first sale. What was the price per bushel?

63. A sold two stoves. On the first he lost \$8 more than on the second; and his whole loss was \$2 less than triple the amount lost on the second. How much did he lose on each?

64. A number of men had done $\frac{1}{4}$ of a piece of work in 6 days, when 12 more men were added, and the job was completed in 10 days. How many men were at first employed?

65. A company discharged their bill at a hotel by paying \$8 each; if there had been 4 more to share in the payment, they would only have paid \$7 apiece. How many were there in the party?

66. In one factory 8 women and 6 boys work for \$72 a week; and in another, at the same rates, 6 women and 11 boys work for \$80 a week. How much does each receive per week?

67. What factor can be removed from $\sqrt{153x}$?

68. Given $\sqrt{x + 12} = \sqrt{a + 12}$, to find x.

69. Given $\dfrac{\sqrt{y}}{y} = \dfrac{y - ay}{\sqrt{y}}$, to find y.

70. Given $\sqrt{x^2 - 4ab} = a - b$, to find x.

71. From a cask of molasses $\frac{1}{3}$ of which had leaked out, 40 liters were drawn, leaving the cask half full. How many liters did it hold?

72. Make a formula to find the per cent commission a factor receives, the amount invested and the commission being given.

73. Divide 20 into two parts, the squares of which shall be in the ratio of 4 to 9.

74. After paying out $\frac{1}{9}$ of my money and then $\frac{1}{8}$ of the remainder, I had \$140 left. How much had I at first?

75. If 1 be added to both terms of a fraction, its value will be $\frac{3}{4}$; and if the denominator be doubled and then increased by 2, the value of the fraction will be $\frac{1}{3}$. Required the fraction.

76. Tiffany & Co. sold a gold watch for \$171, and the per cent gained was equal to the number of dollars the watch cost. Required the cost of the watch.

77. Two Chinamen receive the same sum for their labor; but if one had received \$15 more and the other \$9 less, then one would have had 3 times as much as the other. What did each receive?

78. A drover bought a flock of sheep for \$120, and if he had bought 6 more for the same sum, the price per head would have been \$1 less. Required the number of sheep and the price of each.

79. A certain number which has two digits is equal to 9 times the sum of its digits, and if 63 be subtracted from the number, its digits will be inverted. What is the number?

80. Two river-boatmen at the distance of 150 miles apart, start to meet each other; one rows 3 miles while the other rows 7. How far does each go?

81. A and B buy farms, each paying \$2800. A pays \$5 an acre less than B, and so gets 10 acres more land. How many acres does each purchase?

82. Find a factor that will rationalize $\sqrt{x} + \sqrt{7}$.

83. Find a factor that will rationalize $\sqrt{3x} - \sqrt{3y}$.

84. Given $\sqrt{b^2 + \sqrt{x}} = \dfrac{d + 3}{\sqrt{(b^2 + \sqrt{x})}}$, to find x.

85. The salaries of a mayor and his clerk amount to \$13200; the former receives 10 times as much as the latter. Required the pay of each.

86. What two numbers are those whose sum is to their

difference as 8 to 6, and whose difference is to their product as 1 to 36?

87. What two numbers are those whose product is 48, and the difference of their cubes is to the cube of their difference as 37 to 1?

88. Find the price of apples per dozen, when 2 less for 12 cents raises the price 1 cent per dozen.

89. Two pedestrians set out at the same time from Troy and New York, whose distance apart is 150 miles; one goes at the rate of 8 miles a day, and the other 7. When will they meet?

90. The income of A and B for one month was $1876, and B's income was 3 times A's. Required that of each?

91. A farmer bought a cow and a horse for $250, paying 4 times as much for the horse as for the cow. Find the cost of each.

92. A man rode 24 miles, going at a certain rate; he then walked back at the rate of 3 miles per hour and consumed 12 hours in making the trip. At what rate did he ride?

93. It costs $6000 to furnish a church, or $1 for every square foot in its floor. How large is the building, provided the perimeter be 320 feet?

94. Find 5 arithmetical means between 3 and 31.

95. Find the sum of 50 terms of the series $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, etc.

96. A dealer bought a box of shoes for $100. He sold all but 5 pair for $135, at a profit of $1 a pair. How many pair were there in the box?

97. Two numbers are to each other as 7 to 9, and the difference of their squares is 128. Required the numbers.

98. In a pile of scantling there are 2400 pieces, and the number in the length of the pile exceeds that in the height by 43; required the number in its height and length.

99. Bertha is $\frac{1}{4}$ as old as her mother, but in 20 years she will be $\frac{5}{8}$ as old. What is the age of each?

100. Fifteen persons engage a car for an excursion; but

before starting 3 of the company decline going, by which the expense of each is increased by $1.75. What do they pay for the car?

101. When the hour and minute hands of a clock are together between 8 and 9 o'clock, what is the time of day?

102. A and B wrote a book of 570 pages; if A had written 3 times and B 5 times as much as each actually did write, they would together have written 2350 pages. How many pages did each write?

103. A man and his wife drink a pound of tea in 12 days. When the man is absent, it lasts the woman 30 days. How long will it last the man alone?

104. Find the time in which any sum of money will double itself at 7 per cent simple interest.

105. A purse contains a certain sum, in the proportion of $3 of gold to $2 of silver; if $24 in gold be added, there will then be $7 of gold for every $2 of silver. Required the sum in the purse.

106. A and B in partnership gain $3000. A owns $\frac{5}{8}$ of the stock, lacking $200, and gains $1600. Required the whole stock and each man's share of it.

107. In the choice of a Chief Magistrate, 369 electoral votes were cast for two men. The successful candidate received a majority of one over his rival; how many votes were cast for each?

108. Two ladies can do a piece of sewing in 16 days; after working together 4 days, one leaves, and the other finishes the work alone in 36 days more. How long would it take each to do the work?

109. If a certain number be divided by the product of its two digits, the quotient is $2\frac{1}{4}$; and if 9 be added to the number, the digits will be inverted; what is the number?

110. Find 4 geometrical means between 2 and 486.

111. A trader bought a number of hats for $80; if he had bought 4 more for the same amount, he would have paid $1 less for each; how many did he buy?

112. If the first term of a geometrical series is 2, the ratio 5, and the number of terms 12, what is the last term?

113. A tree 90 feet high, in falling broke into three unequal parts; the longest piece was 5 times the shortest, and the other was 3 times the shortest; find the length of each piece.

114. The sum of 3 numbers is 219; the first equals twice the second increased by 11, and the second equals $\frac{2}{3}$ of the remainder of the third diminished by 19; required the numbers.

115. Required 3 numbers in geometrical progression, such that their sum shall be 14 and the sum of their squares 84.

116. A pound of coffee lasts a man and wife 3 weeks, and the man alone 4 weeks; how long will it last the wife?

117. Two purses contain together $300. If you take $30 from the first and put into the second, each will then contain the same amount; required the sum in each purse.

118. A clothier sells a piece of cloth for $39 and in so doing gains a per cent equal to the cost. What did he pay for it?

119. A settler buys 100 acres of land for $2450; for a part of the farm he pays $20 and for the other part $30 an acre. How many acres were there in each part?

120. What is the sum of the geometrical series 2, 6, 18, 54, etc., to 15 terms?

121. There are 300 pine and hemlock logs in a mill-pond, and the square of the number of pines is to the square of the number of hemlocks as 25 to 49; required the number of each kind.

122. A ship of war, on entering a foreign port, had sufficient bread to last 10 weeks, allowing each man 2 kilograms a week. But 150 of the crew deserted the first night, and it was found that each man could now receive $3\frac{1}{3}$ kilograms a week for the remainder of the cruise. What was the original number of men?

ANSWERS.

INTRODUCTION.

Page 15.

1, 2. Given.
3. 2 cts. A, 6 cts. O.
4. \$8, h.; \$32, c.
5. 9 and 27.
6. $4p$, C; $8p$, B; $16p$, A.
7. $12y$, son; $36y$, father.

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8. \$20, B's; \$80, A's.
9. 15, 30, 45.
10. \$7, calf; \$56, cow.
11. \$5.25, bridle;
\$10.50, saddle;
\$110.25, horse.
12. \$3000, daughter;
\$6000, son;
\$27000, wife.
13. 234, 702, 941.

Page 19.

1–3. Given.
4. $98\frac{1}{2}$.
5. 18.
6. 10.
7. 34.
8. 17.

Page 21.

1. 60.
2. 40.
3. $ac + 8b$.
4. $5b - 2d$.
5. 35.
6. 24.
7. $3x + 2y + ab$.
8. $6b - 7cx + 3a$.
9. $bxy + cxy$.

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10. $\frac{15xy}{2z} + a$.
11. $\frac{b-a}{xy} + 2z$.
12. $3x + xy + 6yz$.
13. $\frac{ax - ay - bx + by}{d}$.
14. 92.
15. 120.

ADDITION.

Page 24.

1, 2. Given.
3. $21ab$.
4. $17xy$.
5. $15a^2$.
6. $-23bcd$.
7. $-16x^3y^2$.
8. $45ab^2$.
9. $-39abx^2y^2$.
10. $29b^2dm^3$.
11. Given.
12. 4.
13. 5.

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14, 15. Given.
16. $8x$.
17. abc.
18. $-12b$.
19. $-12y$.
20. $-2m$.
21. 1.
22. 75.
23. Given.

Page 26.

1. $24a+2b-3d$.
2. $16mn-xy+bc$
3. $16bc+xy-mn$
4. $4ab-3mn+2z$
5. $15xy+ab+b$.
6. Given.
7. $21(a+b)$.
8. $19c(x-y)$.
9. $7a\sqrt{xy}$.
10. $6\sqrt{a}$.
11. $10\sqrt{x-y}$.

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12. Given.
13. $a(7-6b+3d-3m)$.
14. $y(ab+3-2c-5m)$.
15. $m(9+ab-7c+3d)$.
16. $x(13a-3b+c-3d+m)$.
17. $xy(a+b-c)$.
1. Given.
2. 16 cts., b; 30 cts., k.

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3. 26 peaches; 49 pears.
4. 15 and 70.
5. 15 b., 25 g.
6. 1.
7. 7.
8. 7.
9. 356, A; 94, B.
10. 36 and 141.
11. 8.
12. 9 cts., top; 23 cts., ball.
13. \$9, b; \$31, s.
14. 26 cts., A. M.; 74 cts., P. M.
15. Given.
16. 14.
17. 7.
18. 12.
19. 60.
20. 20.

SUBTRACTION.

Page 31.

1, 2. Given.
3. $14xyz$.
4. $-62ab$.
5. $19ab$.
6. $27xy$.
7. $43ac$.
8. $37ax^2$.
9. $51a^2b$.
10. $-44x^2y^2$.

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11. $38a^2b$.
12. 0.
13. $-77m^2x$.
14. $53x^3y$.

15. \$150.
16. 25°.
17. \$420.
18. $4xy-6a$.
19. $13b^2+16am$.
20. $18x^2+y^2+6a$.
21. $13ab+d-x-5m+3n$.
22. $9cd-ab-2m+3n+4y$.
23. $18m-23$.
24. $12x^2-13x$.
25. $16ab+13c+d$.
26. $a-b+c$.
27. $6(a+b)$.
28. $9(a-b+x)$.
29. $5(a+b)$.
30. $-7(x^2-y)$.
31. \$600, A's.
32. 60°.

Page 33.

33. Given.
34. $(2b-c+d)x^2$.
35. $(ab-c-d+x)y$.
36. $a^2(7-b+c)$.
37. $x(ab-3c-d-m)$.
38. $xy(8-ab+c-d)$.
39. $c(2a+bm+d)$.

Page 34.

1–3. Given.
4. $b-c+d-m$.
5. $5x+y-ab+4d$.
6. $2a-b-c+x+y+d$.
7. $a-b+c-a+c+c-a+b$.

MULTIPLICATION.

Page 36.

1–4. Given.
5. $42abc$.
6. $35abcxy$.
7. $8dmxy$.
8. $63bcdxyz$.
9. $56abxy$.
10. $42acdx$.
11. $54bcdm$.
12. $63adfxyz$.

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13. Given.
14. $-45abxy$.
15. $42abcd$.
16. $152abcxy$.
17. $-414abcxy$.
18. $945bcdxy$.

Page 38.

19–21. Given.
22. $15x^3y^2$.
23. $24a^3b^3$.
24. $a^6x^5y^2$.
25. a^2b^{m+n}.
26. $6x^2y^2z$.
27. $18a^5b^4c^2$.
28. 18.
29. 240.
30. $6x^2y$.
31. $-18a^4b^3c$.

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32. $4x^3y^2$.
33. $21a^3b^3$.
34. $40c^4x^ny$.
35. $-28a^4b^3$.
36. $-6x^3y^3$.
37. $21a^2b^2c^4$.
38. $-28a^3c^4$.
39. $x^2y^2z^3$.
1, 2. Given.
3. $6acx^2+8c^2d$.
4. $15a^2b^2x-6acdx+3ax^2$.
5. $-8a^2bd+6ab^2d-2bdm$.
6. $-15a^4c+20a^2b^2c+10a^2c^3$
7, 8. Given.

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1. $6ax+3bx+2ay+by$.

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2. $3ax+4ay$
$-3bx-4by$.
3. $12bd-3cd$
$-4ab+ac$.
4. $6bxy-2ab$
$+6cxy-2ac$.
5. $3ax+4bx-cx$
$-3ay-4by+cy$.
6. $5ax+3ay+az$
$+5bx+3by+bz$.
7. $14cdmx-6abm$
$-21cdnx+9abn$.
8. $24abcx+12mx$
$-32abcy-16my$.
11. $3abc^nxyz^m$.
12. $11abc^3d^{m+n}$.
13. a^2x^{2+n}.
14. $cx(a+b)^4$.
15. $5c(a-b)^5$.
16. $abc(x+y)^{m+n}$.
17. $-3x(a+b)^4$.

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20. a^3+b^3.
21. $a^4+a^2b^2+b^4$.
22. x^4+x^2+1.
23. $3x^4+4x^3y$
$-13x^2-4x^2y^2$
$+22xy-30$.
24. $24a^2x^2-6a^2y^2$.
25. $d^2+bdx+cdx$
$+bcx^2$.
26. Given.
27. x^2-y^2.
28. a^4+a^3+a+1.
29. x^3+3x^2y
$+3xy^2+y^3$.
30. $a^{2m}+2a^mb^n$
$+b^{2n}$.
31. $x^2+2xy+2xz$
$+y^2+2yz+z^2$.

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1. a^2+2a+1.
2. $4a^2+4a+1$.
3. $4a^2-4ab+b^2$.
4. $x^2+2xy+y^2$.
5. $x^2-2xy+y^2$.
6. $1-x^2$.
7. $49y^4-14y^3+y^2$
8. $16m^2-9n^2$.
9. x^4-y^2.
10. $1-49x^2$.
11. $16x^2-8x+1$.
12. $25b^2+10b+1$.
13. $1-2x+x^2$.
14. $1+4x+4x^2$.
15. $64b^2-48ab$
$+9a^2$.
16. $a^2b^2+2abcd$
$+c^2d^2$.
17. $9a^2-4y^2$.
18. x^4-y^2.
19. $x^2-2xy^2+y^4$.
20. $4a^4-x^2$.

Page 45.

1, 2. Given.
3. 36.
4. 24 chickens.
5. Given.
6. 48.
7. 1960.
8. $960.
9. 25.
10. 16.
11. 56.
12. 77.
13. 36 apples.
14. 70 sheep; 100, both.
15. 16 and 12.
16. 24 plums.
17. 42.
18. 144.
19. $21\frac{3}{5}$; $14\frac{2}{5}$.
20. $14\frac{7}{10}$ bu., one; $6\frac{3}{10}$ bu., other.

DIVISION.

Page 47.

1, 2. Given.
3. $2ab$.
4. $5xy$.
5. 5.
6. $2a$.
7. $5ab$.
8. $3bc$.
9. $4mn$.

10, 11. Given.
12. $-3c$.
13. $7b$.
14. $5d$.
15. $-6b$.
16. $7df$.
17. $-9ag$.

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18. Given.
19. d^4.
20. x^6.
21. c^5.
22. z^6.
23. $4b$.
24. $\frac{2x}{y}$.
25. Given.
26. $8abc^2$.
27. $6xz$.
28. $5ab$.
29. $7x^2y$.
30. abc.
31. $2abc$.
32. $8x^3y^3z$.
33. $8a^2bc$.
34. $12d^3x^2y$.
35. $\frac{12x^2}{a}$.
36. $12x^3z^3$.
37. $11m^3n$.

Page 49.

1–3. Given.
4. $b^2+c^3+d^4$.
5. $3x+5$.
6. $3bc-1+4b$.
7. $2by^2-\frac{y}{2}$.
8. $-2x+y$.
9. y^2+z-1.
10. $-5a-4b+6$.
11. $3ab-3a$.
12. $-4x^2-3d^2$
$+ax$.
13. $a^3-5a+2b$.
14. $1+5a-9ad$.
15. $2a-4b-5c$.
16. $2(a+b)^2$
$+3x(a+b)^3$.
17. $9x-9y$.
18. $x(b-c)$
$-a(b-c)$.
19. $3a^2-2a$.
20. $a-a^2+a^3$.

Page 51.

1, 2. Given.
3. $x+y$.
4. $a-b$.
5. $a^2-2ab+b^2$.
6. $c+d$.
7. $x-d$.
8. $2x+3y$.
9. $a-b$.
10. $x+y$.
11. a^2+ab+b^2.
12. $3a+2b$.
13. $a+2$.
14. $a^2-2ax+x^2$.
15. $2x^3+4x^2+8x$
$+16$.
16. $x+5$.
17. $x-2$.
18. $c-x$.
19. $a+b$.
20. $2(a-b)$.
1. 10 yrs., son;
46 yrs., father.
2. 15, F.'s m.;
45, J.'s m.
3. 12 and 60.
4. 12 and 45 p.
5. 31 cts., 1st;
62 cts., 2d;
97 cts., 3d.

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6. 20 cows;
180 sheep.
7. 13½, less;
43½, greater.
8. 9.
9. 5 hours.
10. 8.
11. 7.
12. 5 of each.
13. 4 hours.
14. 33¾.
15. 10.
16. 7 m., A's No.;
14 m., B's;
21 m., C's.

17. \$$2x$, A's m.; \$$4x$, B's m.; \$$8x$, C's m.; \$$14x$, all.

18. 5, 15, and 20.

19. 10, A's; 20, B's; 30, C's.

20. 8, 16, 24.

21. 24.

FACTORING.

Page 54.

1, 2. Given.
3. 2, 3, 3, $aabb$.
4. 2, 2, $5bxxxyy$.
5. 5, 7, $aaabbcc$.
6. 3, 7, $xyyzzz$.
7. $17xxyyyz$.
8. 5, 5, $abbcxxx$.
9. 7, $11aabccd$.
10. 5, 13, $mmnnnx$.

Page 55.

1–3. Given.
4. $b(y + c + 3x)$.
5. $2a(x + y - 2z)$.
6. $3bc(x - 2x - a)$.
7. $8dm(n - 3)$.
8. $7a(5m + 2x)$.
9. $27d(bx - 2my)$.
10. $3a^2(2b + 3c)$.
11. $7axy(3x + 5)$.
12. $5(5 + 3x^2 - 4x^2y^2)$.
13. $x(1 + x + x^2)$.
14. $3(x + 2 - 3y)$.
15. $19a^5(x - 1)$.

Page 56.

1, 2. Given.
3. $(a + b)(a + b)$.
4. $(x - y)(x - y)$.
5. $(m + 2n)(m + 2n)$
6. $(4a + 1)(4a + 1)$.
7. $(7 + 5)(7 + 5)$.
8. $(2a - 3b)(2a - 3b)$.
9. $(y + 1)(y + 1)$.
10. $(1 - c^2)(1 - c^2)$.
11. $(x^m + y^n)(x^m + y^n)$.
12. $(2a^n - 1)(2a^n - 1)$.
13. $(a^2 + b^2)(a^2 + b^2)$.
14. $(ax^2 + y)(ax^2 + y)$.

Page 57.

1. Given.
2. $(a + x)(a - x)$.
3. $(3x + 4y)(3x - 4y)$.
4. $(y + 2)(y - 2)$.
5. $(3 + x)(3 - x)$.
6. $(a + 1)(a - 1)$.
7. $(1 + b)(1 - b)$.
8. $(5a + 4b)(5a - 4b)$.
9. $(2x + y)(2x - y)$.
10. $(1 + 4a)(1 - 4a)$.
11. $(5 + 1)(5 - 1)$.
12. $(x^2 + y^2)(x^2 - y^2)$.
13. $(ax + by)(ax - by)$.
14. $(m^2 + n^2)(m^2 - n^2)$.
15. $(a^m + b^n)(a^m - b^n)$.

Page 59.

1. Given.
2. $(x - 1)(x^2 + x + 1)$.
3. $(x - y)(x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5)$.
4. $(x - 1)(x + 1)$.
5. $(1 - 6y)(1 + 6y)$.
6. Given.
7. $(b - x)(b + x)$.
8. $(d-z)(d^3+d^2z+dz^2+z^3)$.
9. $(a - b)(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5)$.
10. $(x - 1)(x^3 + x^2 + x + 1)$.
11. $(1 - a)(1 + a + a^2 + a^3 + a^4 + a^5)$.
12. $(a - 1)(a^7 + a^6 + a^5 + a^4 + a^3 + a^2 + a + 1)$.
13. Given.

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14. $(x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$.
15. $(a + 1)(a^2 - a + 1)$.
16. $(a + 1)(a^4 - a^3 + a^2 - a + 1)$.
17. $(1 + y)(1 - y + y^2)$.
18. $(1 + a)(1 - a + a^2 - a^3 + a^4)$.
19. $(1 + b)(1 - b + b^2 - b^3 + b^4 - b^5 + b^6)$.

20–28. Given.

DIVISORS.

Page 61.

1, 2. Given.

3. x.
4. b.
5. ac.
6. $2x$.
7. $7m$.
8. $6ab$.

Page 63.

1, 2. Given.

3. $3ac$.
4. $2axy$.
5. $4a^3x^2z^2$.
6. $6ax^3z^2$.

Page 66.

1–5. Given.

6. $x - y$.
7. $a + b$.
8. $b + 2$.
9. $x + 3$.
10. $a - 2$.
11. $a + 3$.
12. $x + 1$.
13. $a - b$.
14. $a - b$.
15. $x^3 + 3x^2 + 3x + 1$.

MULTIPLES.

Page 68.

1, 2. Given.

3. $56a^4b^2c^3d$.
4. $80x^2y^4z^5$.
5. $90a^3b^4c^5$.
6. $420a^4b^4$.
7. $315x^4y^3z^5$.
8. $84m^3n^2y^4$.

Page 69.

9–11. Given.

12. $a^4 + a^3b - ab^3 - b^4$.
13. $x^3 + x^2 - x - 1$.
14. $6a^3 + 11a^2 - 3a - 2$.
15. $m^4 + 2m^3 - m - 2$.

REDUCTION OF FRACTIONS.

Page 74.

1–3. Given.

4. $\frac{1}{3xy}$.

5. $\frac{3ac}{d}$.

6. $\frac{1}{3}$.

7. $\frac{7abc^2}{9x^4y^3}$.

8. $\frac{a-b}{a+b}$.

9. $\frac{x+y}{x-y}$.

10. $\frac{3y-3x}{2x-2z}$.

11. $\frac{1}{x}$.

12. $\frac{1}{x^2+y^2}$.

13. $\frac{x}{a+x}$.

14. $\frac{1}{a-1}$.

15. $\frac{1}{x+y}$.

Page 75.

1. Given.

2. $a-x$.

3. $b-\frac{b^2}{a}$.

4. $b-c$.

5. $b+c+\frac{2c^2}{b-c}$.

6. $a-b$.

7. $b+\frac{2ab}{b-a}$.

8. $a+x+\frac{a}{a-x}$.

9. $3x+1-\frac{3y}{4x}$.

1, 2. Given.

3. $\frac{4xy-b}{y}$.

4. $\frac{10bd+a-c}{2b}$.

5. $\frac{a^2+2ab+b^2+2x}{a+b}$.

6. $\frac{x^3-x}{x+1}$.

7. $\frac{12ac-a+b}{3c}$.

8. $\frac{39x^2+3a}{5x}$.

Page 76.

1. Given.

2. $\frac{12mx}{6m}$.

3. $\frac{24a^2bx}{4ab}$.

4. $\frac{18ac^3+24bc^3}{6c^2}$.

5. $\frac{x^2-y^2}{x+y}$.

6. $\frac{6a^2x^2y-4bx^2y}{3a^2-2b}$.

Page 77.

1, 2. Given.

3. $\frac{ab}{ac}$.

4. $\frac{21a^3}{49a}$.

5. $\frac{x^2-y^2}{x^2-2xy+y^2}$.

6. $\frac{32a^3(x+y)}{8a^2(x+y)^2}$.

Page 78.

1, 2. Given.

3. $\frac{4cx}{4dx}, \frac{4bd}{4dx}, \frac{2d^2x}{4dx}.$

4. $\frac{ac^2y}{2c^2xy}, \frac{2bxy}{2c^2xy}, \frac{2c^2x^2}{2c^2xy}.$

5. $\frac{2a^2+2ab}{3ab+3b^2}, \frac{3bx}{3ab+3b^2}.$

6. $\frac{x^2-2xy+y^2}{x^2-y^2},$

$\frac{x^2+2xy+y^2}{x^2-y^2}.$

7. $\frac{a^2+ab}{3a}, \frac{15a-3}{3a}.$

8. $\frac{bdx}{bd}, \frac{2ad}{bd}, \frac{bc+b}{bd}.$

9. $\frac{2b^2c-2b^2d}{3b^2c-3b^2d}, \frac{3ac-3ad}{3b^2c-3b^2d},$

$\frac{3b^2c+3b^2d}{3b^2c-3b^2d}.$

10. $\frac{2bxy}{2bz}, \frac{bz}{2bz}, \frac{4az}{2bz}.$

11. $\frac{2ax+2ay}{4x+4y}, \frac{12x+12y}{4x+4y},$

$\frac{4x^2+4y^2}{4x+4y}.$

12. $\frac{a^2-2ax+x^2}{a^2-x^2},$

$\frac{a^2+2ax+x^2}{a^2-x^2}.$

Page 79.

1. Given.

2. $\frac{2acx}{4bcx}, \frac{4b^2c^2}{4bcx}, \frac{bxy}{4bcx}.$

3. $\frac{3c^2d}{3abc}, \frac{2bcx}{3abc}, \frac{3bxy}{3abc}.$

4. $\frac{6ay}{12y}, \frac{4by}{12y}, \frac{3cy}{12y}, \frac{12x}{12y}.$

5. $\frac{4abc^2}{4b^2c}, \frac{8cd}{4b^2c}, \frac{bx^2y}{4b^2c}.$

6. $\frac{16a^2b}{24a^2c}, \frac{18a^2c}{24a^2c}, \frac{24x}{24a^2c},$

$\frac{3a^2c}{24a^2c}.$

7. $\frac{ac}{2bc}, \frac{2cd}{2bc}, \frac{2xy}{2bc}.$

8. $\frac{(a+b)^2}{a^2-b^2}, \frac{(a-b)^2}{a^2-b^2}, \frac{a^2+b^2}{a^2-b^2}.$

9. $\frac{4xy\,(x+y)}{6xy\,(x+y)}, \frac{6a\,(x+y)}{6xy\,(x+y)},$

$\frac{abxy}{6xy\,(x+y)}.$

10. $\frac{ad}{a^2b^2}, \frac{bx}{a^2b^2}.$

11. $\frac{b^2cdx}{ab^2c^2d}, \frac{acdm}{ab^2c^2d}, \frac{ab^2y}{ab^2c^2d}.$

12. $\frac{x^2z}{xy^2z}, \frac{ayz+byz}{xy^2z}, \frac{dy^2}{xy^2z}.$

13. $\frac{4cmx^2+4cnx^2}{12a^2cx^2},$

$\frac{6acm-6acn}{12a^2cx^2}, \frac{3a^2m^2x}{12a^2cx^2}.$

ADDITION OF FRACTIONS.

Page 80.

1, 2. Given.

3. $\frac{27ac}{2xy}$.

4. $\frac{39dxz}{5abc}$.

5. $\frac{5b}{x}$.

6. $\frac{7a + 2b}{c}$.

7. Given.

Page 81.

8. $\frac{2x^2 + 2y^2}{xy}$.

9. $\frac{45a + 12x + 20y}{60}$.

10. $\frac{8ax + 6 + 9ay}{12a}$.

11. $\frac{ab - ac + bx + cx}{b^2 - c^2}$.

12. $\frac{3x - y}{2xy}$.

13. $\frac{2a + 2ax + 3}{ay}$.

14. $\frac{ax - ay + abx + aby}{x^2 - y^2}$.

15. $\frac{5cd^2 + 30xy + 3bdx^2}{15dx}$.

16. $\frac{3ah - 2dn - d^2}{3dh}$.

17. $\frac{am - dy}{my}$.

18. $\frac{nx - mx - hy}{my - ny}$.

19. -6.

20. $\frac{4adx + 6bcx - bdm}{bdx}$.

1. Given.

2. $a + c + \frac{bx + 2d}{2x}$..

3. $x + \frac{am - ay - bx + bd}{bm - by}$.

4. $3d + a + b - c - \frac{xy + z}{2}$.

5. $5x + \frac{2a - by}{2b}$.

Page 82.

6. Given.

7. $\frac{3bd + 2a}{b}$.

8. $\frac{a + b - 4cy}{c}$.

9. $\frac{x + y - a^2}{a}$.

10. $\frac{3x^2 - 2xy - y^2 + a - b}{x - y}$.

11. $\frac{x - y - a^2 + 6ab - 5b^2}{a - b}$.

12. $\frac{2x^2 + 2xy - 2x - 2y + a + b}{x - 1}$

SUBTRACTION OF FRACTIONS.

Page 83.

1, 2. Given.

3. $\frac{9abc}{d}$.

4. $\frac{17xyz - 5cd}{a}$.

5, 6. Given.

7. $\frac{ay - dm + bm}{my}$.

8. $\frac{by - dy + bm}{my}$.

9. $\frac{17d - 9a}{12}$.

Page 84.

10. $\frac{hy + hm + dm}{my}$.

11. $\frac{h - my}{y}$.

12. $a + \frac{bd + ch}{cd}$.

13. $\frac{6a + 5b - 2d}{6}$.

14. $\frac{ad + ay - bc + cx}{bd - dx + by - xy}$.

15. $a - \frac{2x + 3dy}{2y}$.

16. $\frac{x^2 - y^2 - 10a + 10b}{10x + 10y}$.

17. $\frac{4x - y + 3c + 6a}{6}$.

MULTIPLICATION OF FRACTIONS.

Page 85.

1–4. Given.

5. $h + 3d$.

6. $\frac{ab}{4}$.

7. $\frac{6cx - 9cy + 4dx - 6dy}{15c + 4d}$.

8. $2abc$.

9. $\frac{a + b}{4 + 5y}$.

10. $\frac{2a^2 + 2a^2b}{b + 1}$.

11. $8x^2 + 12x$.

12. $2ax - 2bx + 3a - 3b$.

13. abc.

14. $\frac{a + b}{5}$.

15. $\frac{2x + y}{5}$.

16. $\frac{9c - 3d}{4}$.

17. $3xy + 3x$.

18. $\frac{m^2}{x - z}$.

Page 87.

1, 2. Given.

3. $6xy$.

4. $\frac{dx}{ay}$.

5. $\frac{x^2 - y^2}{y^2z + yz^2}$.

6. $\frac{3}{8a + 24x}$.

7. $\frac{4hy}{3cx}$.

8. $\frac{d(a + b)}{cx}$.

9, 10. Given.

11. $\frac{y - 3x}{2xy}$.

12. $\frac{b - 2a}{3ab}$.

13. $\frac{a^2x + a^2 + ab}{by}$.

14. $\frac{xy + 2x + y^2 + 2y}{xy}$.

15. $\frac{x^4 - y^4}{x^2y}$.

16. $2b + 4$.

Page 88.

1. Given.

2. $\frac{abdx}{y}$.

3. $\frac{abd + acd}{xy}$.

4. $\frac{mx + nx}{4}$.

5. $\frac{4ac + 4ch}{d}$.

6. $\frac{15ax - 5xy}{y}$.

7. $\frac{5bx^2 + 5b}{x - 1}$.

8. $7x - 7ax$.

9. $\frac{acx - acy}{3}$.

10. $\frac{3ac}{2}$.

11. $\frac{2ax^3 + 2ax}{3x - 3}$.

12. $\frac{8x^2y}{a + b}$.

13. $\frac{6am}{x + 1}$.

14. $\frac{2abxy + b^2xy}{4a + b}$.

15. $1 - n$.

1. $\frac{9cx - 3dx}{4}$.

2. $3x(y + 1)$.

3. $\frac{xy + 2x + y^2 + 2y}{xy}$.

4. $9x$.

5. $\frac{a}{6}$.

6. x.

7. $\frac{m^2}{x - z}$.

8. $6a^2y^2$.

9. $x^2 - y^2$.

10. $\frac{2x + y}{5}$.

11. $\frac{x^4 - y^4}{x^2y}$.

12. $2b + 4$.

13. $2a(c + d)$.

14. $\frac{y - 3x}{2xy}$.

15. b^2.

16. $\frac{2(x + y)}{x}$.

DIVISION OF FRACTIONS.

Page 90.

1–4. Given.

5. $\frac{2x}{n}$.

6. $\frac{2a}{3bc}$.

7. $1+\frac{b}{c}$.

8. $\frac{a+y}{x}$.

9. $\frac{a}{2b}$.

10. $\frac{a^2+ac+c^2}{b+c}$.

11. $\frac{x+y}{a+c}$.

12. $\frac{x+2y}{a^2-b^2}$.

Page 92.

1. Given.

2. 3 times.

3. 5.

4. Given.

5. $\frac{a^2by}{cdx}$.

6. $\frac{ax^2}{4y^3}$.

7. $\frac{3x}{2y}$.

8. $\frac{2}{x-1}$.

9. $\frac{a^2-a}{2}$.

10. $\frac{2x^2y}{3}$.

11. 6.

12. $\frac{a^2b}{2a+2b}$.

13. $\frac{3x+3y}{abcxy}$.

14. $\frac{x-a}{6bc^2}$.

15. $\frac{8y^2}{3ax+3ay}$.

16. $\frac{4xy^2}{3az+3bz}$.

17. $\frac{bx^3}{a^3}$.

18. $\frac{4dy}{b}$.

19. $\frac{b}{4dy}$.

20. $\frac{4dy}{b}$.

1. Given.

2. $\frac{abdmy}{cx}$.

3. $\frac{mx+nx}{4}$.

4. $\frac{ax+x^3}{5c}$.

5. $\frac{25x^3-5xy}{y}$.

6. $\frac{5ax^2+5a}{x+1}$.

7. $3x-3ax$.

Page 93.

8, 9. Given.

10. $\frac{a^2+2a+1}{a^2-2a+1}$.

11. $\frac{a^2-b^2}{x^2-y^2}$.

12. $\frac{x^3-xy^2+x^2y-y^3}{a-b}$.

13. $\frac{x^2-y^2}{a^2-b^2}$.

Page 94.

1. $\frac{5a}{4xyz}$.

2. $\frac{35bcd}{243xy^2}$.

3. $\frac{24xy}{cd}$.

4. $\frac{3a}{x+y}$.

5. $\frac{23xz}{17y(a+b)}$.

6. $\frac{3b(x - y)}{c}$.

7. Given.

8. $\frac{a}{a + x}$.

9. $\frac{4c}{c + 2}$.

10. $\frac{4d(d^2 - x^2)}{3a(c^2 - x^2)}$.

11. $\frac{2c}{a^2 - ac + c^2}$.

12. $\frac{4(a^2 - 2ax + x^2)}{3x}$.

13. $\frac{1}{(a + b)^2}$.

14. $\frac{x}{x^2 + 2x + 1}$.

15. $\frac{c}{x}$.

SIMPLE EQUATIONS.

Page 97.

1, 2. Given.

3. $a - b + c - d$.

4. $a + b - ab + c$.

Page 98.

5. Given.

6. $2 - a + b$.

7. $b + c - a - 3$.

8. $ad - bc + 2m - 8$.

9. $17 - 3ab - d$.

10. $4cd + d - 3bh - 1$.

11. $32 - c + d$.

12. 11.

Page 100.

1, 2. Given.

3. 15.

4. 12.

5. $24\frac{4}{7}$.

6. Given.

7. $\frac{5bd - 4ad}{3b}$.

8. $\frac{10a - 2b - c}{16}$.

9. $7\frac{1}{2}$.

10. $\frac{16c}{15}$.

11. 12.

12. 24.

13. 14.

14. $-1\frac{3}{35}$.

15. Given.

Page 101.

1. 8.

2. 50.

3. 30.

4. 9.

5. 7.

6. 9.

7. 72.

8. 30.

9. 5.

10. 28.

11. 12.

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16. 12.

17. 60.

18. 4.

19. 20.

20. $ab + ac$.

21. $\frac{dn}{a}$.

22. $\frac{6c}{3a + 2b}$.

23. $\frac{ad - bc}{ac}$.

24. $\frac{2ab}{ac - 2c}$.

25. $\frac{5a + 13b}{24}$.

26. $\dfrac{24b + 150c - 20a}{45}$.

27. $\dfrac{3a - 6}{4}$.

28. $\dfrac{1}{2a - 1}$.

29. $\dfrac{a^2(c - a + c^2 - ac)}{c^2}$.

30. $-\dfrac{b^2}{2b} = -\dfrac{b}{2}$.

31. $\dfrac{a - 1}{2}$.

32. $7b$.

33. 84.

Page 103.

34. 20.

35. 24.

36. 1.

37. $1\frac{5}{9}$.

38. $16\frac{1}{6}$.

39. $1\frac{1}{23}$.

40. 36.

41. 11.

42. 1200.

43. $14\frac{1}{7}$.

44. 5.

45. $4\frac{4}{7}$.

46. $\dfrac{b}{2}(1 - a^2)$.

47. $\dfrac{2a - 2b + c}{2b}$.

48. $\dfrac{3a - 6}{4}$.

49. $\dfrac{6d}{5b}$.

50. $\dfrac{1 - 8a}{1 + 8a}$.

51. $\dfrac{a}{4} - 2$.

Page 105.

1. Given.
2. \$8, vest;
   \$32, coat.
3. \$1500, A;
   \$3000, B;
   \$4500, C.
4. 40 men;
   80 boys;
   880 women.
5. 40 miles;
   80 miles.
6. $133\frac{1}{3}$ barrels.
7. 12 p., 1st;
   24 p., 2d;
   60 p., 3d.
8. $28\frac{4}{5}$ feet.
9. \$120.
10. \$50, B's sh.;
    \$100, A's sh.;
    \$150, C's sh.
11. 18 yrs., w.;
    36 yrs., m.
12. \$3000.
13. 16 and 41.
14. \$6000.

Page 106.

15. 13.
16. 30. days.
17. 240 m., one;
    180 m., other.
18. 12 in., one;
    16 in., other.
19. \$25, H.;
    \$175, C.
20. 8h. 24m. A. M.
21. $9\frac{3}{13}$ days.
22. 32 of each.
23. 30, 75, and 45.
24. 25 cts., ch.;
    75 cts., goose;
    \$1.50, turkey.
25. 8 ft. 8 in.
26. 40 and 60.
27. $\dfrac{ad}{c + d}$, less.
    $\dfrac{ac}{c + d}$, greater.
28. \$1128.

Page 107.

29. Given.
30. 60 lbs., b.;
120 lbs., m.
31. 15 yrs., B's;
30 " A's.
32. $32\frac{1}{3}$ yrs., C's;
$37\frac{1}{3}$ " B's;
$40\frac{1}{3}$ " A's.
33. 1175 votes d.;
1325 " s.
34. 164 artillery;
472 cavalry;
564 infantry.
35. \$$533\frac{1}{3}$, B's;
\$$633\frac{1}{3}$, A's;
\$$833\frac{1}{3}$, C's.
36. \$56.25, one;
\$93.75, other.

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37. \$336, pr. one.
\$280, " other
38. 14 yrs., y'ngest;
16 " next;
18 " eldest.
39. $16\frac{2}{3}$ days.
40. 550.
41. 30 and 18.
42. $\frac{12a}{13}$.
43. 225 acres, A;
315 " B.
44. $2\frac{6}{7}$ hrs.
45. 5, 1st part;
8, 2d "
2, 3d "
24, 4th "
46. 9.
47. 47 sheep.
48. \$120.

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49. 60 min.
50. $\frac{d}{m-n}$.
51. 300 leaps.
52. $\frac{ab}{a-c}$, one.
$\frac{bc}{a-c}$, other.
53. 72 lbs.
54. 36 hours;
312 miles.
55. 20 yrs., s.;
40 yrs., f.
56. 280.
57. \$324, 1st;
\$108, 2d;
\$144, 3d.
58. Given.

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59. 8, 1st part;
12, 2d "
16, 3d "
60. 9 in. and 12 in.
61. \$75.
62. 27 days.
63. \$1575, one;
\$2625, other.
64. 12 days.
65. \$720.
66. \$384, sum;
\$162, A's sh.;
\$118, B's "
\$104, C's "
67. Given.

Page 111.

68. 6 and 8.
69. 3456, one;
2304, other.
70. 3 m. an hour.
71. 400 in.,
or $33\frac{1}{3}$ ft.
72. 8 k. of one name
6 k. of another;
3 k. " "
2 k. " "
73. 7 and 8.
74. 240 leaps of d.
75. 100 days;
30000 m., 1st;
24000 m., 2d;

TWO UNKNOWN QUANTITIES.

Page 114.

1. Given.
2. $x=8$, $y=4$.
3. $x=12$, $y=6$.
4. $x=18$, $y=2$.
5. $x=1$, $y=3$.
6. $x=16$, $y=35$.
7. $x=3$, $y=2$.
8. Given.
9. $x=4$, $y=5$.
10. $x=6$, $y=12$.
11. $x=5$, $y=6$.
12. $x=10$, $y=3$.
13. $x=11$, $y=9$.
14. $x=3$, $y=2$.

Page 116.

15, 16. Given.
17. $x=3$, $y=5$.
18. $x=4$, $y=7$.
19. $x=7$, $y=2$.
20. $x=16$, $y=35$.
21. $x=3$, $y=2$.
1. $x=4$, $y=5$.
2. $x=8$, $y=2$.
3. $x=5$, $y=3$.
4. $x=3$, $y=4$.
5. $x=3$, $y=4$.
6. $x=12$, $y=3$.
7. $x=3$, $y=5$.
8. $x=4$, $y=3$.

Page 117.

9. $x=34$, $y=46$.
10. $x=4$, $y=2$.
11. $x=16$, $y=7$.
12. $x=8$, $y=1$.
13. $x=60$, $y=36$.
14. $x=10$, $y=20$.
15. $x=5$, $y=2$.
16. $x=2$, $y=4$.
17. $x=8$, $y=6$.
18. $x=4$, $y=9$.
19. $x = 6$, $y = 12$.
20. $x=18$, $y=14$.
1. $x=43$, $y=27$.
2. 4 cts., lemons; 6 cts., oranges.
3. 233 v.; 142 v.
4. 21 and 54.
5. \$48, cow; \$96, horse.
6. 40 l.; 50 g.

Page 118.

7. 3 and 2.
8. 11111 m., one; 9999 m., other.
9. 56.
10. \$320, B's; \$250, A's.
11. $\frac{4}{15}$.
12. \$900, A's; \$2400, B's.
13. 31 and 17.
14. \$6000 h.; \$2500 g.
15. 30 and 20.
16. \$560, B's; \$720, A's.
17. 25 y. and 35 y.
18. \$180, 1st; \$115, 2d.

Page 119.

19. 140 m., ship; 160 m., steamer
20. 12 and 18.
21. 108 ft.
22. 30 yrs.; 13 verses.
23. 10 l.; 30 g.
24. 3 oxen; 21 colts.
25. 53.
26. \$5000, B's cap.; \$4800, A's "
27. \$21 or 63 g.
28. $x = \dfrac{an}{n+1}$, $y = \dfrac{a}{n+1}$.

THREE OR MORE UNKNOWN QUANTITIES.

Page 121.

1. Given.
2. $x=7$, $y=5$, $z=4$.
3. $x=2$, $y=3$, $z=5$.
4. $x=8$, $y=4$, $z=2$.
5. $x=4$, $y=3$, $z=5$.
6. $x=24$, $y=6$, $z=23$.
7. $x=7$, $y=10$, $z=9$.
8. $x=24$, $y=60$, $z=120$.

9–11. Given.

12. $w=2$, $x=3$, $y=4$, $z=5$.
13. $x=2$, $y=3$, $z=4$.

Page 123.

1. 12 yrs., 1st;
   15 " 2d;
   17 " 3d.
2. \$5, s.; \$4, l.;
   \$20, c.
3. 5, 8, and 11.
4. 630 men, 1st;
   675 " 2d;
   600 " 3d.
5. 50 cts., 1st;
   60 " 2d;
   80 " 3d.
6. 105 min., A;
   210 " B;
   420 " C.
7. 18 = 1st;
   22 = 2d;
   10 = 3d;
   40 = 4th.
8. 46 m., A's;
   9 " B's;
   7 " C's.
9. \$64, A's;
   \$72, B's;
   \$84, C's.

GENERALIZATION.

Page 125.

1. 3 chickens.
2. 30 rods.
3. 12.
4. $51\frac{1}{10}$ yrs.
5. 7 ft.
6. 34.
7. \$205, \$187.

Pages 127, 128.

8. \$961, A's;
   \$614, B's.
9. 1248, g.;
   902, l.
10. $4\frac{4}{9}$ days.
11. $5\frac{5}{8}$ hrs.
12. $22\frac{2}{13}$ days.
13. \$67.32.

Page 129.

14. 9077.
15. \$1036.12½.
16. 587.19 bu.
17. $12\frac{4}{9}$ per cent.
18. 40 per cent.
19. 60 per cent.
20. \$3000.

Pages 130–133.

21. \$37500.
22. \$31250.
23. \$2700, B's;
    \$2300, C's.
24. \$5625.
25. 1842½ A.
26. \$55.80.
27. \$190.32.
28. \$1253.
29. \$418.60.
30. \$5250.
31. \$3865.86.
32. \$1339.29.
33. \$222.22 +.
34. 2½ yrs.
35. Given.
36. 3h. $16\frac{4}{11}$m. P.M.
37. 6h. $32\frac{8}{11}$m. P.M.
38. 9h. $49\frac{1}{11}$m. P.M.

INVOLUTION.

Page 137.

2. $a^2b^2c^2$.
3. $a^2b^2c^3$.
4. $x^3y^3z^3$.
5. $a^5b^5c^5$.
6. $16x^8y^4$.
7. $216a^9b^6$.
8. $625a^{12}b^8c^4$.
9. $64a^{12}b^6c^{12}$.
10. $a^8b^8c^8d^8$.
11. $x^ny^nz^n$.

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12. $(a+b)^{10}$.
13. $(a+b)^{2n}$.
14. $(x-y)^{mn}$.
15. $(x+y)^{2n}$.
16. $(a^3+b^3)^2$.
17. $a^9b^6h^{12}$.
19. $\dfrac{27a^3b^6}{8a^3}$.
20. $\dfrac{16a^8b^4c^{12}}{x^8y^{12}}$.
21. $\dfrac{49a^{2n}b^4}{9a^4b^{2n}}$.
22. $\dfrac{2^m}{a^m}$.
23. $\dfrac{a^{mn}b^{n\times n}}{x^ny^{n\times n}}$.
26. $x^3+6x^2y+6x^2+12xy^2+24xy+12x+8y^3+24y^2+24y+8$.

Page 142.

1. $a^4+4a^3b+6a^2b^2+4ab^3+b^4$.
2. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$.
3. $c^7+7c^6d+21c^5d^2+35c^4d^3+35c^3d^4+21c^2d^5+7cd^6+d^7$.
4. $x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$.
5. $x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7$.
6. $y^{10}+10y^9z+45y^8z^2+120y^7z^3+210y^6z^4+252y^5z^5$ $+210y^4z^6+120y^3z^7+45y^2z^8+10yz^9+z^{10}$.
7. $a^9-9a^8b+36a^7b^2-84a^6b^3+126a^5b^4-126a^4b^5+84a^3b^6$ $-36a^2b^7+9ab^8-b^9$.
8. $m^{11}+11m^{10}n+55m^9n^2+165m^8n^3+330m^7n^4+462m^6n^5$ $+462m^5n^6+330m^4n^7+165m^3n^8+55m^2n^9+11mn^{10}+n^{11}$.
9. $x^{12}-12x^{11}y+66x^{10}y^2-220x^9y^3+495x^8y^4-792x^7y^5$ $+924x^6y^6-792x^5y^7+495x^4y^8-220x^3y^9+66x^2y^{10}$ $-12xy^{11}+y^{12}$.
10. $a^n+na^{n-1}b+n\dfrac{n-1}{2}a^{n-2}b^2+n\dfrac{n-1}{2}\times\dfrac{n-2}{3}a^{n-3}b^3$

$$+n\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}a^{n-4}b^4+\text{etc.}$$

13. $x^3 + 3x^2 + 3x + 1$.
14. $b^4 - 4b^3 + 6b^2 - 4b + 1$.
15. $1 - 5a + 10a^2 - 10a^3 + 5a^4 - a^5$.
16. $1 + na + n\frac{n-1}{2}a^2 + n\frac{n-1}{2} \times \frac{n-2}{3}a^3 +$ etc.

Pages 143, 144.

17. $x^3+3x^2y+3x^2z+3xy^2+6xyz+3xz^2+y^3+3y^2z+3yz^2+z^3$.
18, 19. Given.
20. $x^2 + 2x(y+z) + y^2 + 2yz + z^2$.
21. $a^2 - 2a(b-c) + b^2 - 2bc + c^2$.
22. $a^2 + 2a(x+y+z) + x^2 + 2x(y+z) + y^2 + 2yz + z^2$.
23. Given.
24. $\frac{9a^2 + 12a + 4}{9}$.
25. $\frac{4a^2 - 4ac + c^2}{4}$.
26. $\frac{36 - 168abc + 196a^2b^2c^2}{49}$.
27. $\frac{b^2 - 6bmxy + 9m^2x^2y^2}{m^2}$.
28, 29. Given.

MULTIPLICATION AND DIVISION OF POWERS.

Pages 145, 146.

1, 2. Given.
3. a^8.
4. x^{-8}.
5. b^{-3}.
6. a^{m+n}.
7. $a^{-7}b^5$.
8. $a^8c^4d^8$.
9. $b^2c^3y^2$.
10. $a^3y^{-9}z^7$.
11. Given.
12. a^{11}.
13. x^{-11}.
14. b^2.
15. c^{-2}.
16. $x^7y^{-8}z^5$.
17. $4abc^{-4}$.
18. $3x^6y$.
19. $12a^5c^8$.
20–23. Given.
24. $\frac{a}{x^3y}$.
25. $\frac{ay^{-4}}{b}$.
26. $\frac{a}{d^5x^2}$.
27. $\frac{bx^{-n}}{a}$.

EVOLUTION.

Pages 148, 149.

13. $a^{\frac{4}{3}}$.
14. $x^{\frac{5}{4}}$.
15. $y^{\frac{12}{8}}$.
16. $a^{.25}$.
17. $a^{.2}$.
18. $a^{.6}$.
19. $b^{.8}$.
20. $x^{1.4}$.
21. $y^{.75}$.
22–25. Given.

Pages 151, 152.

1, 2. Given.

3. a^3.

4. $a^{\frac{1}{4}}$.

5. $4^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}$.

6. $2a^{\frac{5}{3}}b^2$.

7. $3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$.

8. $2a^m$.

9. $3^{\frac{1}{3}}a^{\frac{2}{3}}x^2$.

10. $6a^2b$.

11. $2^{\frac{1}{3}}x^{\frac{2}{3}}y^{\frac{2}{3}}$.

12. $8a^{\frac{2}{3}}b^3$.

13. $(13)^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}$.

14. $7x^2y^3$.

15. $3a^{\frac{2}{3}}b^{\frac{2}{3}}$.

16. $\dfrac{7x^2}{8y}$.

1. Given.

2. $x + 2$.

3. $a - 1$.

4. $1 + x$.

5. $x + \frac{2}{3}$.

6. $a - \frac{1}{2}$.

7. $x + \dfrac{b}{2}$.

Page 153.

8. Given.

9. $x + y + z$.

10. $a - 2b + 1$.

11. $a^2 + 2b - 2$.

12. $1 - 2b^2 + x$.

13. $2a^2 - 4a + 2$.

14. $a - \dfrac{b}{2}$.

15. $\dfrac{x}{y} - \dfrac{y}{x}$.

RADICAL QUANTITIES.

Page 156.

1–5. Given.

6. $a\sqrt{b}$.

7. $2a\sqrt{2b}$.

8. $6\sqrt{xy}$.

9. $6\sqrt[3]{3}$.

10. $15\sqrt[3]{5}$.

11. $36a\sqrt{7b}$.

12. $3a\sqrt[3]{2c}$.

13. $21a\sqrt{1 - 3b}$.

14. $4x\sqrt[3]{y}$.

15. $3a\sqrt[4]{b}$.

16. $6a\sqrt{13c}$.

17. $12a\sqrt{11}$.

Page 157.

1. Given.

2. $\sqrt[3]{8a^9b^3}$.

3. $\sqrt{(2a + b)^2}$.

4. $\sqrt{(a - 2b)^2}$.

5. $\sqrt{9a^2b}$.

6. $\sqrt[3]{8a^7b}$.

7. $\sqrt[4]{16x^2y^{\frac{4}{3}}z}$.

8. $\sqrt[3]{\dfrac{a^3b^3c^3}{27}}$.

9. $\sqrt[3]{27\,(a - b)^3}$.

10. $\sqrt[3]{a^6}$.

11. $\sqrt[4]{a^{12}c^8}$.

12. $\sqrt{(a - b)^2}$.

13. $\sqrt[n]{a^{mn}}$.

Pages 158, 159.

1. Given.

2. $(a^3)^{\frac{1}{6}}$, $(b^4c^4)^{\frac{1}{6}}$.

3. $9^{\frac{1}{4}}$, $(125)^{\frac{1}{4}}$.

4. $(a^3)^{\frac{1}{6}}$, $(1296)^{\frac{1}{6}}$.

5. $\sqrt[12]{15625}$, $\sqrt[12]{81}$, $\sqrt[12]{8}$.

6. $\sqrt[6]{4x^8}$, $\sqrt[6]{125x^9}$.

7. $\sqrt[6]{64a^9}$, $\sqrt[6]{4a^4}$.

8. $\sqrt[mn]{a^n}$, $\sqrt[mn]{b^m}$.

9. $(b^n)^{\frac{1}{2n}}$, $(c^2)^{\frac{1}{2n}}$.

10. $\sqrt[6]{(a + b)^3}$, $\sqrt[6]{(a - b)^2}$.

11. $\sqrt[12]{(x - y)^8}$, $\sqrt[12]{(x + y)^9}$.

1, 2. Given.

3. $(3^{\frac{3}{2}})^{\frac{1}{6}}$, $(4^2)^{\frac{1}{6}}$.

4. $(a^{10})^{\frac{1}{5}}$, $(b^{15})^{\frac{1}{5}}$.

5. $(a^4)^{\frac{3}{4}}$, $(b^{\frac{4}{9}})^{\frac{3}{4}}$.

6. $(a^{\frac{4}{5}})^{\frac{5}{8}}$, $(b^{\frac{16}{15}})^{\frac{5}{8}}$.

7. $(a^{\frac{n}{m}})^{\frac{1}{n}}$, $(b^{\frac{2n}{m}})^{\frac{1}{n}}$.

ADDITION OF RADICALS.

Page 160.

1–3. Given.

4. $5\sqrt{3}$.

5. $2\sqrt{5}+4\sqrt{3}$.

6. $(2b+3a)\sqrt{b}$.

7. $(a^2+3c)\,3ab$.

8. $(9x+8a)\sqrt{2a}$.

9. $25\sqrt[3]{2}$.

10. $118\sqrt{3}$.

11. $30a\sqrt{b}$.

12. $(5bx+6x^2)\sqrt{c}$.

13. $4x^2\sqrt[3]{y}$
 $+5x\sqrt{xy}$.

SUBTRACTION OF RADICALS.

Page 161.

1. Given.

2. $8\sqrt{7}$.

3. $4\sqrt{30}-12\sqrt{7}$.

4. $52\sqrt{5}$.

5. $(21x^2-10)\sqrt{ax}$

6. $2\sqrt[3]{a+b}$.

7. $7\sqrt[3]{b}$.

8. $9b\sqrt[3]{2bx}$.

9. $a^{-\frac{1}{3}}$.

10. $1\frac{5}{6}\sqrt{3}$.

MULTIPLICATION OF RADICALS.

Page 162.

1–3. Given.

4. $90\sqrt{10}$.

5. abx.

6. $\sqrt{a^2-b^2}$.

7. $\sqrt{acxy}$.

8. $a\sqrt{ac}$.

9. Given.

10. $\sqrt[mn]{a^nx^m}$.

11. $42\sqrt[3]{2}$.

12. $12a$.

13. 6.

14. $4ax$.

15. Given.

16. $\sqrt{5}$.

17. $2\sqrt{5}$.

18. $(m+n)\sqrt[3]{m+n}$

19. $a\sqrt{\dfrac{3d}{c}}$.

DIVISION OF RADICALS.

Page 164.

4. $\sqrt{3a^3}$, or
 $a\sqrt{3a}$.

5. $3\sqrt{bx}$.

6. $(a^2+x)^{\frac{1}{6}}$.

7. $12\,(ay)^{\frac{1}{6}}$.

8. $3b\sqrt{x}$.

9. $\dfrac{15a}{2}\sqrt{b}$.

10. $2a\sqrt{x}$.

11. $(a+b)^{\frac{1}{n}}$.

12. $15x\sqrt{x}$.

13. $\sqrt[3]{x-y}$.

14. $16\sqrt{2}$.

15. 32.

INVOLUTION OF RADICALS.

Page 164.

1, 2. Given.

3. $a^{\frac{3}{4}}$.

4. $18x$.

5. $8a$.

6. $\frac{x^4}{4}\sqrt{2x}$.

7. $8ax^3\sqrt{a}$.

8. $9b^2$.

9. $a^2+2a\sqrt{y}+y$.

EVOLUTION OF RADICALS.

Page 165.

1. Given.

2. $3\sqrt[3]{a}$.

3. $2\sqrt[6]{3x}$.

4. $\sqrt[6]{9xy}$.

5. $\sqrt[6]{8b^3}$.

6. $\sqrt[6]{a^2bc}$.

7. $\sqrt[3]{\frac{4}{9}}$.

8. $ac^{\frac{1}{6}}$.

9. $2a^{\frac{1}{48}}$.

10. $a^{\frac{1}{8}}b^{\frac{1}{24}}$.

11. $\sqrt{2a}$.

12. $a^{\frac{1}{n}}b^{\frac{1}{6n}}c^{\frac{1}{6n}}$.

Page 166.

1–3. Given.

4. $a^{\frac{1}{3}}$.

5. $a^{\frac{1}{3}}c^{\frac{2}{3}}$.

6. $(a+b)^{\frac{2}{3}}$.

7. $\sqrt{ac}$.

8. $\sqrt[3]{x+y}$.

9. $\sqrt[4]{a+b}$.

10. $\sqrt{a+b+c}$.

Page 167.

1, 2. Given.

3. $x-4\sqrt{9}$.

4. 3.

5. $\sqrt{7}-\sqrt{a}$.

6. 31.

7. $\sqrt{3a}+\sqrt{3b}$.

8. $a-5$.

9. $3\sqrt{a}-\sqrt{8}$.

10. $4\sqrt{2a}+5\sqrt{b}$.

Page 168.

1–3. Given.

4. $\frac{c\sqrt[3]{x^2}}{x}$.

5. $\frac{a\sqrt[3]{c^2}}{c}$.

6. $\frac{x+2\sqrt{xy}+y}{x-y}$.

7. $\frac{x(\sqrt{a}+\sqrt{c})}{a-c}$.

8. $\frac{\sqrt{3}-1}{2}$.

9. $\frac{\sqrt{3}+1}{2}$.

RADICAL EQUATIONS.

Pages 169, 170.

1–3. Given.

4. $(d-a-c)^n$.

5. 25.

6. $4\frac{25}{36}$.

7. 256.

8. 1100.

9. 31.

10. 21.

11. 252.

12. $\frac{b^6-2ab^3+a^2-b}{2a}$.

13. Given.

14. $\frac{1}{1-a}$.

15. $a\sqrt{\frac{1}{3}}$.

16. Given.

17. 4.

18. $\frac{9}{20}$.

19. $\frac{1}{1-a}$.

PURE QUADRATICS.

Page 173.

1. Given.
2. $x = \pm 5$.
3. $x = \pm 3$.
4. $x = \pm 4$.
5. $x = \pm 5$.
6. $x = \pm 4$.
7. $x = \pm 6$.
8. $x = \pm 4$.
9. $x = \pm \sqrt{6}$.
10. $x = \pm 7$.
11. $x = \pm 2$.
12. $x = \pm a$.
13. $x = \pm 1$.
14. $x = \pm 2$.
15. $x = \pm 3$.
16. $x = \pm 1$.
17. $x = \pm \frac{1}{a - 1}$.
18. Given.
19. $x = \pm 3$.
20. $x = \pm 2a$.
21. $x = \pm \sqrt{c^2 + d^2}$
22. $x = \pm \frac{1}{2}$.
23. $x = \pm \sqrt{a^2 + b^2}$
24. $x = \pm 26$.

Page 174.

1. $x = \pm 36$.
2. $x = \pm 8$.
3. 40 rods.
4. 30 rods.
5. 12, one; 30, other.
6. \$6.
7. 80.
8. 15, less. 60, greater.
9. 27 yds.; \$1.50, price.
10. $x = \pm 16$.
11. 77 ft.
12. $x = \pm 16$.

AFFECTED QUADRATICS.

Pages 178, 179.

1–5. Given.
6. $x = 6$ or 2.
7. $x = 9$ or -1.
8. $x = 3a \pm \sqrt{d + 9a^2}$.
9. Given.
10. 15 or -4.
11. 20 or -7.
12. Given.

1. Given.
2. 10 or -7.
3. 2 or -5.
4. 3 or $1\frac{2}{3}$.
5. 4 or $-1\frac{3}{4}$.
6. $x = 2$.
7. $-\frac{a}{2b} \pm \sqrt{ab + d + \frac{a^2}{4b^2}}$.
8. $-2a \pm \sqrt{b + 4a^2}$.
9. 7 or -5.
10. $3 \pm 2\sqrt{-1}$.
11. 6 or -3.
12. 2 or -3.
13. $\frac{b}{2c} \pm \sqrt{bd - ch + \frac{b^2}{4c^2}}$

Page 181.

1, 2. Given.
3. 3 or $-4\frac{1}{3}$.
4. 5 or -6.
5. $\frac{1}{3}$ or -2.
6. 2 or $-\frac{1}{4}$.
7. 4 or $-4\frac{2}{3}$.
8. 4 or -1.
9. 3 or $-4\frac{2}{3}$.
10. 9 or 6.
11. 4 or $-3\frac{2}{3}$.

Page 182.

1. 3 or 1.
2. 4 or 1.
3. 3 or $\frac{1}{2}$.

4. 2 or -12.
5. $1\frac{1}{2}$ or $\frac{2}{3}$.
6. 11 or 3.
7. $1\frac{2}{3}$ or $-1\frac{1}{2}$.
8. $-\frac{1}{2}$ or $-1\frac{23}{50}$.
9. 4 or $2\frac{1}{12}$.
10. $1 \pm \sqrt{-a^2+1}$.
11. $-m$
 $\pm \sqrt{b^2+m^2}$.
12. 1 or $-1\frac{1}{8}$.
13. 1 or -28.
14. 10 or $-\frac{7}{10}$.
15. $-\frac{1}{6}$ or $-\frac{1}{2}$.
16. 4 or -1.
17. 4 or $-1\frac{2}{3}$.
18. 5 or $-4\frac{4}{7}$.
19. $1\frac{1}{2}$ or $-\frac{1}{2}$.
20. 4 or -1.
21. $1\frac{1}{2}$ or $-\frac{1}{2}$.
22. $4\frac{1}{2}$ or $\frac{1}{2}$.
23. 3 or $-1\frac{1}{3}$.
24. 4 or $-1\frac{3}{4}$.
25. $\frac{3}{5}$ or $\frac{1}{5}$.
26. $\frac{1}{2}$ or -1.
27. $n \pm m$.
28. $3b$ or $3a-3b$.

Page 183.

1–3. Given.
4. ± 2 or $\pm \sqrt{2}$.
5. $\pm \sqrt{3}$ or
 $\pm \sqrt{-1}$.
6. $\sqrt[8]{7} = 1.91+$.
7. $\frac{1}{4}$ or $-\frac{3}{4}$.
8. $\frac{1}{8}$ or -8.
9. $4\frac{1}{4}$ or $\frac{1}{4}$.
10. 4 or $-21\frac{1}{3}$.

Pages 184, 185.

1. 8 or 4, one;
 4 or 8, other.
2. \$60 or \$40.
3. 6 or 4, one;
 4 or 6, other.
4. 16s., \$5 each.
5. 5 or $-6\frac{1}{2}$.
6. 16 scholars.
7. \$30 or \$20;
 \$20 or \$30.
8. 60 or 40, one;
 40 or 60, other.
9. 36 rds. length;
 28 " breadth
10. 20 in file;
 80 in rank.
11. 10 lambs.
12. 2 and 2.
13. 4 and 1.
14. 121 yds. long;
 120 " wide.
15. 6 m., A's rate;
 5 m., B's rate.
16. 120, A;
 80, B.
17. 42 and 6.
18. 4 lemons, A;
 6 " B.
19. 14 ft., length;
 10 " breadth.
20. 12 rows;
 15 trees in each
21. 52.
22. 20 persons.

Page 186.

23. 8 or -10, less;
 15 or -12, gr.
24. 16 and 20.
25. 50 and 25.
26. 121 and 25.
27. 12 ft., fore-w.;
 15 ft., hind-w.
28. 2 or -18, one;
 18 or -2, oth.
29. $\frac{5}{6}$ and $\frac{1}{2}$.
30. 3, less;
 18, greater.
31. 16 or 36 yrs.
32. 28 rods, length;
 20 " breadth.
33. 15 yrs., A's;
 8 yrs., B's.
34. 20 lbs. pepper.

Pages 188, 189.

1. Given.
2. $x = 4$ or 3;
 $y = 3$ or 4.
3. $x = 7$ or 5;
 $y = 5$ or 7.
4. $x = 8$, $y = 6$.
5. $x = 10$ or -12;
 $y = 12$ or -10.
6. $x = 10$;
 $y = 12\frac{19}{47}$ or 7.

7. $x = 9$;
$y = 4$ or $\frac{268}{293}$.

9. $x = 5$ or 4;
$y = 4$ or 5.

10. $x = 5$ or -3;
$y = 3$ or -5.

11. $x = 3$; $y = 2$.

12. $x = \pm 5$;
$y = \pm 3$.

Page 190.

15. $x = 15$ or 12;
$y = 12$ or 15.

16. $x = 21$ or -7;
$y = 7$ or -21.

17. $x = 625$;
$y = 16$.

18. $x = 2$ or 1;
$y = 1$ or 2.

Page 191.

1. 8 or -4, gr.;
4 or -8, less.

2. 30 yrs., wife;
31 " man.

3. $x = 9\sqrt{2}$, gr.;
$y = \pm\sqrt{2}$, less.

4. 40 rows;
25 trees in each.

5. 40 yds, length;
24 " breadth.

6. 9 and 3.

7. 11 and 7.

8. 31 rds., length;
19 " width.

9. ± 7 and ± 4.

10. 25 m. and 23 m.

11. 12 and 4.

12. 3 or -2, one;
2 or -3, other.

RATIO.

Page 195.

3. $\frac{1}{4}$.
4. $\frac{1}{8}$.
5. $\frac{1}{5}$.
6. $2a$.
7. $3c$.
8. 10.
9. $\frac{1}{8}$.
10. $\frac{1}{4}$.
11. $\frac{a}{2}$.
12. $x - y$.
14. $\frac{4}{9}$.
15. $\frac{2}{3x}$.
17. $\frac{1}{4}$.
18. $\frac{1}{121}$.
19. Equality.
20. Equality.
21. Gr. inequality.
22. Less inequal'y.
23. $\frac{38}{19} > \frac{15}{9}$.
24. $\frac{8}{25} < \frac{10}{25}$.
25. 7.
26. 98.

PROPORTION.

Page 204.

2. 4.
3. 6400.
4. 12.
5. 32 and 24.
6. 10 and 8.
7. 16 and 12.
8. 6 and 4.
9. 48 and $9\frac{3}{5}$.
10. 430 r., length;
320 r., breadth.
11. 20 r.; 30 r.
12. 9 and 15.
13. 20 and 16.

ARITHMETICAL PROGRESSION

Page 207.

2. 9.
3. 68.
4. -5.
5. $1\frac{3}{4}$.
6. .91.
8. 43.
9. 15.
10. $44\frac{1}{2}$.
11. $49x$.
12. $3an - a$.

Page 208.

2. $762\frac{1}{2}$.
3. 216.
4. 1400.
5. $25\frac{1}{2}$.
6. 610.
7. 175.
8. 810.

Page 209.

1. 58.
2. 278.
3. 11.
4. — 43.
5. $2\frac{1}{3}$.
6. $-\frac{21}{25}$.
7. 1024.
8. 192.

Page 210.

1. 175.
2. 1130.
3. 6.
4. 6.
5. 259.
6. 13.
7. — 11.
8. 0.
9. 255.
10. 62.
11 61.

Page 212.

1. 1, 7, 13, 19, 25, 31.
2. 3, $7\frac{1}{2}$, 12, $16\frac{1}{2}$, 21, $25\frac{1}{2}$, 30, $34\frac{1}{2}$, 39, $43\frac{1}{2}$, 48.
1. 47.
2. — 6.
3. 102.
4. 2, $11\frac{2}{3}$, $21\frac{1}{3}$, 31, $40\frac{2}{3}$, $50\frac{1}{3}$, 60.
5. $1683\frac{1}{3}$.
6. $98\frac{3}{19}$.
7. 5776.
8. 10100.

Page 213.

9. 5.
10. 6, $13\frac{1}{3}$, $20\frac{2}{3}$, 28, $35\frac{1}{3}$, $42\frac{2}{3}$, 50, $57\frac{1}{3}$, $64\frac{2}{3}$, 72.
11. 12, 21.6, 31.2, 40.8, 50.4, 60, 69.6, 79.2, 88.8, 98.4, 108.
12. 975.
14. 3, 5, and 7.
15. 10100 yards, or $5\frac{3}{4}$ mi., nearly.

Page 214.

16. 156.
17. \$62.50.
18. \$667.95.
19. 300.
20. \$1.20, int.; \$2.20, amt.
21. 20, 40, and 60.
22. 16.61 + days.
23. 30, 40, 50, 60.
24. 3 days.
25. 140.
26. \$178, last pay't; \$5370, debt.

GEOMETRICAL PROGRESSION.

Page 216.

1. 160.
2. 4374.
3. $4\frac{1}{2}$.
4. 320.
5. 112.
6. — 31250.

Pages 217-221.

2. 2498.
3. $5554\frac{5}{9}$.
4. $33331\frac{2}{3}$.
5. 21.
6. $7\frac{2}{3}$.
1. 30000.
2. 15625.
3. 2.
4. 3.
5. 6.
6. 5.
1. 161.
2. 2.
3. $128\frac{4}{5}$.
4. 5.

5. 179.
6. 180.7.
2. $\frac{1}{2}$, 2, 8, 32, 128.
1. 4371.
2. $\frac{1640}{2187}$.
3. 7174453.
4. $2\frac{173051}{177147}$.
5. 9565938.

6. 43046721.
8. $4095.
9. $196.83, l. c.; $295.24, wh. c.
10. $10.23.
11. 2, 6, 18.
12. $4294967.295.
13. 10, 30, 90, 270.

Page 223.

14. $120, $60, $30.
15. 3, 15, 75, 375, 1875.
16. $108, $144, $192, $256.
17. $\frac{11}{10}$, or 1.1.
18. 8, 4, 2, 1.

INFINITE SERIES.

Page 231.

1. $1\frac{1}{2}$.
2. $\frac{2}{3}$.
3. 1.
4. $1\frac{1}{2}$.
5. 2.
6. 9.
7. 10.
8. $\frac{1}{3}$.
9. $\frac{3}{8}$.
10. $\frac{1}{a-1}$.
11. 50 rods.

LOGARITHMS.

Pages 235-237.

2. 1537.93.
3. 1.973.
4. 111
6. 78.
7. .0375.
8. 14.38.
9. 2.723.
10. 2906.3.
11. .1814.
12. — 4.619.

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14. .0003321.
15. 33.335.
16. 191.77.
18. 5.23.
19. 1.0836.
20. 2.504.
21. 2.124.

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23. .342+.
24. .546+.
25. .324+.
26. Given.

BUSINESS FORMULAS.

Pages 245-260.

2. $349.60.
4. $16\frac{2}{3}$ per cent.
6. $12600.
8. $840.
10. $600.
12. $16\frac{2}{3}$ years.
13. 10 years.
15. $2\frac{1}{2}$ per cent.
17. $2010.14.
19. $5414.28.
21. $2769.23, pr. w.; $830.77, disc.
22. $6000, pr. w.; $1800, disc.
24. $1718.75.
26. $2125.
28. 2.33\frac{1}{3}$.
29. $8.83+.
31. $5\frac{255}{309}$ per cent.
32. $12\frac{1}{2}$ per cent.
33. $9\frac{8}{13}$ per cent.
34. O. 8 per cents.
36. $24630.54, in.; $369.46, com.
38. $1332.
39. $6290.15.

40. \$905.80.
41. \$3278.69.
42. \$2278.48.
44. \$6130.67.
45. \$2767.60.
47. \$2336.25.
49. \$14166.67.
51. \$14775.
53. \$249.77.

Pages 265–268.

2. $+\sqrt{xy}$.
3. -6.
4. $6\sqrt{-1}$.
5. $\sqrt{-xy}$.
7. 1.
8. $\sqrt{\frac{-x}{y}}$.
9. $\sqrt{\frac{x}{-y}}$.
10. $5\sqrt{2}$.
11. $\frac{c}{d}$.
2. Impossible.
3. Impossible.

TEST EXAMPLES.

Page 274.

1. $75a$.
2. $57x + 7$.
3. $3ax + 3ab + 2cd$.
4. $7bc + 3cd - 10xy + 5mn$.
5. ——.
6. ——.
7. ——.
8. 8.
9. 31.
10. $c(3b^2 - 6b^2c - cd)$.
11. $3x^2(y - 3z - 6yz)$
12. $(a^n + b^n)(a^n - b^n)$
13. $2 \times 2(2a - 1)$.
14. $(a^2 + 1)(a + 1) \times (a - 1)$.
15. 20, 11.
16. ——.
17. 24 shots.

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18. $\pm\sqrt{ab}$; $\pm\sqrt{\frac{a}{b}}$.
19. $\frac{a+1}{b}$.
20. $\frac{1}{a-b}$.
21. $(3xy + 2z) \times (3xy + 2z)$.
22. $(3b - c)(3b - c)$.
23. ——.
24. 640 rods.
25. $29\frac{1}{4}$ miles.
26. $\frac{b-1}{b+1}$.
27. ——.
28. ——.
29. 1.
30. $\frac{a+b}{a-b}$.
31. $\frac{3a^2 - a^4}{1 - a^4}$.
32. 120.
33. \$50.

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34. 6 hours.
35. $x = 5$; $y = 3$.
36. $x = 5$; $y = 2$.
37. ——.
38. $85\frac{1}{3}$ miles.
39. $\frac{12}{18}$.
40. $8\frac{3}{4}$ g.; $5\frac{1}{4}$ l.
41. $x = 4$; $y = 6$; $z = 8$.
42. $x = 2$; $y = 4$; $z = 8$.
43. $x =$ \$40 A, $y =$ \$60 B, $z =$ \$80 C.
44. 3 meters f. w.
    6 " h. w.
45. 8 ft. one; 10 ft.
46. ——.
47. 49 and 77.
48. 48 meters.
49. \$2.46 a meter.
50. ——.
51. \$100 horse, \$200 carriage.

52. ——.
53. 65 hectares y.
100 " elder.
54. 26.
55. \$5 l.; \$6 s.
56. 577 v.; 848 v.
57. 90 yrs. A; 45 y. B; 15 y. C.
58. $9\sqrt{3}$.
59. $y\sqrt{1+a}$.

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60. $(x^4)^{\frac{1}{4}}, (y^8)^{\frac{1}{8}}$.
61. $\sqrt[3]{27(a-b)^3}$.
62. 90 cts.
63. \$10; \$18.
64. 15 men.
65. 28 persons.
66. \$4 b. \$6 w.
67. 9.
68. a.
69. $\dfrac{1}{1-a}$.
70. $a+b$.
71. 240 liters.
72. ——.
73. 12 and 8.

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74. \$180.
75. $\frac{8}{11}$.
76. \$90.
77. \$21.
78. 24 s. \$5 pr.
79. 81.
80. 45 m.; 105 m.
81. 80 A; 70 B.
82. $\sqrt{x}-\sqrt{7}$.
83. $\sqrt{3x}+\sqrt{3y}$.
84. $d^2+6d-2b^2d$
$+9-6b^2+b^4$.
85. \$12000 Mayor;
\$1200 Clerk.
86. 216 g.; $30\frac{6}{7}$ l.

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87. 8 and 6.
88. 8 cts.
89. 10 days.
90. \$1407 B;
\$469 A.
91. \$50 cow;
\$200 h.
92. 6 miles.
93. 100 ft. × 60 ft.
94. $7\frac{2}{3}$, $12\frac{1}{3}$, 17,
$21\frac{2}{3}$, $26\frac{1}{3}$.
95. $637\frac{1}{2}$.
96. 50 pair.
97. 18 and 14.
98. 32 h.; 75 l.
99. 10 y. B;
30 y. M.
100. \$105.

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101. 8 : $43\frac{7}{11}$ o'clk.
102. 250 pp. A;
320 " B.
103. 20 days.
104. $14\frac{2}{7}$ yrs.
105. \$30.
106. \$9000 whole s.
\$4800 A;
\$4200 B.
107. 184 v. and
185 v.
108. 24 d.; 48 d.
109. 45.
110. 6, 18, 54, 162.
111. 16 hats.

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112. 97656250.
113. 10 ft.; 30 ft.;
50 ft.
114. 95; 42; 82.
115. 2, 4, 8.
116. 12 weeks.
117. \$180; \$120.
118. \$30.
119. 45 A.; 55 A.
120. 14348906.
121. 125 p. 175 h.
122. 375 men.